J. DIFFERENTIAL GEOMETRY
31 (1990) 215–248

# EINSTEIN METRICS ON PRINCIPAL TORUS BUNDLES

McKENZIE Y. WANG & WOLFGANG ZILLER

## 0. Introduction

The goal of this paper is to describe a new class of Einstein metrics and discuss their geometrical and topological properties.

The building blocks for these examples are Kähler-Einstein manifolds with positive first Chern class. Let $(M, g)$ be such a manifold. Then there exists a positive integer $q$ such that $c_1(M) = q\alpha$ and $\alpha$ is an indivisible class in $H^2(M;\mathbb{Z})$.

**Theorem 1.** *Let $(M_i, g_i)$, $i = 1, \cdots, m$, be Kähler-Einstein manifolds with $c_1(M_i) > 0$ and $c_1(M_i) = q_i\alpha_i$ with $\alpha_i$ indivisible. Let $P$ be the total space of a principal torus bundle over $B = M_1 \times \cdots \times M_m$ whose characteristic classes in $H^2(B;\mathbb{Z})$ are integral linear combinations of $\alpha_1, \cdots, \alpha_m$. Then $P$ carries an Einstein metric with positive scalar curvature iff $\pi_1(P)$ is finite.*

The condition that $\pi_1(P)$ be finite is necessary for the existence of an Einstein metric with positive scalar curvature by the theorem of Bonnet and Myers. In the special case of circle bundles we get

**Corollary 1.** *Every nontrivial circle bundle over $M_1 \times \cdots \times M_m$ whose Euler class is an integral linear combination of $\alpha_1, \cdots, \alpha_m$ carries an Einstein metric with positive scalar curvature.*

In the case of these circle bundles we also show that the Einstein metric we obtained is uniquely determined up to scaling by the property that the projection $P \to M_1 \times \cdots \times M_m$ is a Riemannian submersion with totally geodesic fibers and that the metric on the base is a product of Kähler-Einstein metrics. In general, the metric on the base will not be Einstein.

In the special cases of circle bundles over $P^1\mathbb{C} \times P^2\mathbb{C}$ and over $P^1\mathbb{C} \times P^1\mathbb{C} \times P^1\mathbb{C}$, these Einstein metrics were discovered independently by the physicists R. D'Auria, Castellani, Fré, and van Nieuwenhuizen [3], [7] in

Received February 1, 1988. The first author was partially supported by a University Research Fellowship from the Natural Sciences and Engineering Research Council of Canada. The second author was partially supported by a grant from the National Science Foundation and would like to thank the Institut des Hautes Études Scientifiques and the Max Planck Institute for their hospitality.

an attempt to construct a Kaluza-Klein supergravity theory in dimension 11.

If $m = 1$, one obtains the existence of an Einstein metric on the unit circle bundle of the canonical line bundle over a Kähler-Einstein manifold with $c_1 > 0$, which is due to S. Kobayashi [16].

Until recently, the only known Kähler-Einstein manifolds with $c_1 > 0$ were the compact homogeneous Kähler-Einstein manifolds, which occur as adjoint orbits of a compact Lie group [2, Chapter 8]. These include the compact hermitian symmetric spaces, and, in particular, $P^n\mathbb{C}$, the most fundamental example. In [25], [17], Koiso and Sakane constructed the first nonhomogeneous Kähler-Einstein manifolds with $c_1 > 0$, which are certain $S^2$ bundles over products of Kähler-Einstein manifolds. Recently Tian [28] proved that there are Kähler-Einstein metrics on the Fermat hypersurfaces in $P^{n+1}\mathbb{C}$ of degree $n$ and $n+1$ (which have positive first Chern class), and Tian and Yau [29] constructed Kähler-Einstein metrics on the del Pezzo surfaces $P^2\mathbb{C}\# k(-P^2\mathbb{C})$ for $3 \le k \le 8$. In both cases, except for the del Pezzo surface with $k = 3$, the isometry group is finite.

Apart from the above Kähler-Einstein manifolds, the only known nonhomogeneous compact Einstein manifolds with positive scalar curvature are the cohomogeneity one examples constructed by Bérard Bergery [1] (see also [24]), which are not Kähler.

In our examples, if each $(M_i, g_i)$ is a homogeneous Kähler-Einstein manifold, then the Einstein metric on $P$ is also homogeneous. We shall see that even these new homogeneous Einstein metrics have many interesting properties. More generally, we prove in §3

**Theorem 2.** *With $M_i$ and $P$ as in Theorem* 1, *we have* $\mathrm{coh}(P) = \sum_{i=1}^m \mathrm{coh}(M_i)$, *at least if the topology of P is sufficiently complicated.*

For circle bundles, the condition on the topology of $P$ is that $\sum_{i=1}^m b_i^2$ should be sufficiently large, where the Euler class $e(P) = \sum_i b_i\alpha_i$. We suspect that Theorem 2 actually holds in almost all cases. Using the above nonhomogeneous Kähler-Einstein manifolds we obtain

**Corollary 2.** *There exist Einstein manifolds with positive scalar curvature and any given cohomogeneity.*

If we use the examples of Tian and Yau, we obtain

**Corollary 3.** *There exist odd-dimensional Einstein manifolds with positive scalar curvature and one-dimensional isometry group.*

Einstein metrics are the critical points of the total scalar curvature functional on the space of metrics with volume 1. Hence it is interesting to examine the set of Einstein constants for Einstein metrics of volume 1. We will prove

**Theorem 3.** *For a fixed base $M_1 \times \cdots \times M_m$, if we normalize the Einstein metrics on the total spaces of the principal torus bundles in Theorem* 1 *to have volume one, then for any sequence of principal torus bundles, no two of which are equivalent by an automorphism of the fiber, the sequence of Einstein constants converges to* 0.

In §2 we examine the topology of the total spaces of some of these torus bundles. In general, for two different torus bundles over $M_1 \times \cdots \times M_m$, the total spaces have different homotopy types. Hence one gets, in all but finitely many dimensions $\geq 7$, infinitely many homotopy types for Einstein manifolds with positive scalar curvature, which can be chosen to be either all homogeneous or all nonhomogeneous. The only previously known examples of this type were the seven-dimensional homogeneous manifolds $\mathrm{SU}(3)/S^1$ (see [30]).

We will examine the total spaces $M_{k,l}^{p,q}$ of the circle bundles over $P^p\mathbb{C} \times P^q\mathbb{C}$ with Euler class $e = k\alpha_1 + l\alpha_2$, where $\alpha_1$, $\alpha_2$ are generators in $H^2$ of each factor. One easily sees that $M_{k,l}^{p,q}$ is simply connected iff $(k, l) = 1$, which we will assume from now on. We show that

(a) If $2 \leq p < q$, then $M_{k,l}^{p,q}$ is homeomorphic (or diffeomorphic) to $M_{k',l'}^{p,q}$ iff $|k| = |k'|$ and $|l| = |l'|$.

(b) $M_{k,l}^{p,p}$ all have the same integral cohomology ring, but if $p \geq 4$ then $M_{k,l}^{p,p}$ is homeomorphic (or diffeomorphic) to $M_{k',l'}^{p,p}$ iff $(|k'|, |l'|)$ is a permutation of $(|k|, |l|)$.

However, in the remaining cases, i.e., $1 = p \leq q$, the total spaces are frequently diffeomorphic. More precisely,

(a) $M_{k,l}^{1,1}$ is diffeomorphic to $S^2 \times S^3$ for all $k, l$.

(b) $M_{k,\pm 1}^{1,q}$, $q > 1$, is diffeomorphic to $S^2 \times S^{2q+1}$ if $q$ is odd or if $q$ and $k$ are even, and diffeomorphic to the nontrivial $S^{2q+1}$ bundle over $S^2$ if $q$ is even and $k$ is odd.

(c) $M_{k,\pm 2}^{1,q}$, $q > 1$, is diffeomorphic to one of the three nontrivial $P^{2q+1}\mathbb{R}$ bundles over $S^2$. The bundle type is independent of $k$ if $q$ is odd and depends on the parity of $k$ if $q$ is even.

(d) More generally, if we fix $q$ and $|l|$, then there are only finitely many diffeomorphism types among the manifolds $M_{k,l}^{1,q}$.

Together with Theorem 3, we obtain

**Corollary 4.** *On $S^2 \times S^{2q+1}$, the nontrivial $S^{4q+1}$ bundle over $S^2$, and some nontrivial $P^{2q+1}\mathbb{R}$ bundles over $S^2$, there are infinitely many nonisometric Einstein metrics with positive scalar curvature.*

Moreover, since on the moduli space of Einstein metrics of volume 1 on a fixed manifold the scalar curvature functional is constant on each component, it follows from Theorem 3 that

**Corollary 5.** *The moduli space of Einstein metrics for the manifolds in Corollary* 4 *has infinitely many components.*

These are the first examples of a moduli space with infinitely many components. In the cases of negative and zero Einstein constant no such examples are known. In general, the moduli space of Einstein metrics on a compact manifold is known to have at most countably many components, while the moduli space for Kähler-Einstein metrics can have at most finitely many components. We do not know whether any of the components of the moduli spaces for the manifolds in Corollary 4 contain more than one point.

In the examples of Tian and Yau, except for the del Pezzo surfaces with $k = 3$ or 4, one obtains nontrivial deformations of the Kähler-Einstein metrics from nontrivial deformations of the complex structures. For our principal circle bundles over products of these manifolds, the deformations on the base always give rise to deformations of the Einstein metrics on the total space. In general, we do not know if these deformations are trivial or not. However, using the proof of Theorem 2, we can at least show that

**Corollary 6.** *If $P$ is the total space of a circle bundle as in Corollary* 1 *with Euler class $e(P) = \sum_i b_i \pi_i^* \alpha_i$, then if $\sum_i b_i^2$ is sufficiently large, any nontrivial deformation of the Kähler-Einstein metric $g_i$ on $M_i$ gives rise to a nontrivial deformation of the corresponding Einstein metric on $P$.*

But we were not able to find examples among our manifolds whose moduli space of Einstein metrics contains infinitely many components of positive dimension.

On p. 19 of [2] the question was raised whether the total scalar curvature functional satisfies a Palais-Smale condition in the sense that a sequence of metrics whose deviation from being Einstein goes to zero must contain a subsequence converging to an Einstein metric. Our examples show that this is in general false, since the sequence of Einstein metrics on any of the manifolds in Corollary 4 cannot have a convergent subsequence.

Next we observe that the total spaces $P$ of the circle bundles over $P^p\mathbb{C} \times P^q\mathbb{C}$ are in fact homogeneous spaces and each Einstein metric from Corollary 1 is homogeneous. When infinitely many of these total spaces are diffeomorphic or homeomorphic, we will see that the transitive actions are in general all inequivalent. More specifically, we have

**Corollary 7.** *For the manifolds $P$ in Corollary* 4, *there exist infinitely many isomorphic but nonconjugate maximal compact subgroups of* $\mathrm{Diff}(P)$

*(resp.* Homeo$(P)$) *that act transitively on P. Moreover, these subgroups are the connected isometry groups of the Einstein metrics on P coming from Corollary* 1.

Previously no examples were known where a given Lie group acts transitively on a manifold in more than one way.

After we obtained the topological conclusions mentioned above about the $M_{k,l}^{p,q}$ we asked a number of topologists for the homeomorphism and diffeomorphism classifications of these manifolds. M. Kreck and S. Stolz [18] succeeded in doing this for the circle bundles over $P^1\mathbb{C} \times P^2\mathbb{C}$. The surprising fact is that the homeomorphism and diffeomorphism classifications do not agree. For example, if $l \equiv 0$ (4), $l \equiv 0, 3, 4$ (7), and $l \neq 0$, then $M_{k,l}^{1,2}$ is homeomorphic to $M_{k',l}^{1,2}$ iff $k \equiv k'$ $(2l^2)$ and diffeomorphic to $M_{k',l}^{1,2}$ iff $k \equiv k'$ $(2 \cdot 28l^2)$. Moreover, in this particular case the number of smooth structures on these manifolds is the maximum possible 28 and they are all obtained via connected sum with exotic 7-spheres.

This gives a surprising answer to a problem posed by W. C. Hsiang and W. Y. Hsiang [11], who asked whether two homogeneous manifolds that are homeomorphic must be diffeomorphic.

Combining the results of Kreck and Stolz with ours, one gets

**Corollary 8.** *There exist manifolds P in dimension 7 such that for each exotic 7-sphere* $\Sigma$, $P\#\Sigma$ *admits infinitely many nonisometric Einstein metrics which lie in different components of the moduli space of Einstein metrics. Furthermore, each* $P\#\Sigma$ *admits infinitely many inequivalent transitive actions by the same compact Lie group, and the above Einstein metrics have these subgroups of* Diffeo$(P\#\Sigma)$ *as connected isometry groups.*

Finally, we examine our examples in two other contexts. We will show that if one considers the total spaces of a sequence of our principal torus over a fixed base and normalizes the Einstein metric so that the Einstein constant is 1, then this sequence of total spaces collapses in the sense of Gromov and converges in the Hausdorff topology to the base manifold. The limit metric on the base is a product of the Kähler-Einstein metrics but in general will not be Einstein. We will also examine our example in connection with a pinching theorem of Cheeger [4], [5] for symmetric spaces.

## 1. Existence of Einstein metrics

We first discuss the Einstein condition for a general class of metrics on principal bundles, and then specialize to torus bundles.

Let $\pi\colon P \to B$ be a smooth principal $G$-bundle where $G$ is a connected Lie group acting on $P$ on the right. Given a Riemannian metric $(\ ,\ )_B$ on $B$, a left-invariant metric $(\ ,\ )_G$ on $G$, and a principal connection $\theta$ for $\pi\colon P \to B$, one can define $(X, Y)_P = (\pi_* X, \pi_* Y)_B + (\theta(X), \theta(Y))_G$. This makes $\pi$ into a Riemannian submersion with totally geodesic fibers isometric to $G$ with the above left-invariant metric. Conversely, if $\pi\colon P \to B$ is a Riemannian submersion with totally geodesic fibers, then all fibers of $\pi$ are isometric. If the fiber metric is a left-invariant metric on $G$, then $(\ ,\ )_P$ arises exactly by the above construction. Note that $G$ acts via isometries of $(\ ,\ )_P$ iff $(\ ,\ )_G$ is a bi-invariant metric.

We will denote by $\mathscr{H}$ and $\mathscr{V}$ respectively the horizontal and vertical distributions of the Riemannian submersion $\pi\colon P \to B$. In general, $X$, $Y$, $Z$ will denote horizontal tangent vectors on $P$ and $U$, $V$, $W$ vertical tangent vectors. Let $\Omega$ be the curvature form of the principal connection $\theta$. Then the metric $(\ ,\ )_P$ is Einstein with constant $E$ iff

(1.1′) $\theta$ is Yang-Mills with respect to $(\ ,\ )_B$ and $(\ ,\ )_G$,

$$\mathrm{Ric}_G(\theta(U), \theta(V)) + \frac{1}{4}\sum_{i,j}(\Omega(X_i, X_j), \theta(U))_G(\Omega(X_i, X_j), \theta(V))_G$$

(1.2′) $$= E(\theta(U), \theta(V))_G,$$

(1.3′) $$\mathrm{Ric}_B(\pi_* X, \pi_* Y) - \frac{1}{2}\sum_i (\Omega(X, X_i), \Omega(Y, X_i))_G = E(\pi_* X, \pi_* Y)_B,$$

where $\{X_i\}$ is an orthonormal basis of horizontal tangent vectors at the point in question. Notice that if $(\ ,\ )_G$ is bi-invariant, then (1.3′) becomes an equation on $B$, and (1.2′) is invariant under the right action of $G$. (1.1′) is equivalent to $\mathrm{Ric}_P(\mathscr{V}, \mathscr{H}) \equiv 0$. The above equations follows from O'Neill's formulas for a Riemannian submersion [21], and more details can be found in [2, pp. 236–250]. The O'Neill tensor $A$ in our situation is related to $\Omega$ by $\theta(A_X Y) = -\frac{1}{2}\Omega(X, Y)$.

In general, the existence of a Yang-Mills connection is already a difficult problem, and (1.2′), (1.3′) are even more restrictive, since they couple together the metrics on $B$, $G$, and the connection $\theta$. The only previously known solutions of these equations include (i) bi-invariant and left-invariant Einstein metrics on certain compact Lie groups $G$ viewed as principal $H$-bundles over $G/H$, where $H$ are suitable closed subgroups of $G$ (see [13], [6], [31]), and (ii) Einstein metrics on principal circle bundles over Kähler-Einstein manifolds with positive scalar curvature whose Euler class is a rational multiple of the first Chern class of the base (see [16]).

In (ii) notice that there is a unique simply connected circle bundle, and all other circle bundles are covered by it.

Solution (ii) suggests a natural simplification of (1.1′)-(1.3′) by assuming that $G$ is an $r$-dimensional torus $T$. Then, any left-invariant metric is bi-invariant and flat, so $\mathrm{Ric}_G \equiv 0$. But more importantly, and curvature $\omega = d\theta$ is the pull-back of a closed 2-form $\eta$ on $B$, and the difficult Yang-Mills condition is replaced by the harmonicity of $\eta$. More precisely, the Einstein conditions become

(1.1) $\eta$ is a harmonic $\mathbb{R}^2$-valued form on $B$ with respect to $(\ ,\ )_B$,

(1.2) $$\frac{1}{4}\sum_{i,j}(\eta(X_i,X_j),U)_G(\eta(X_i,X_j),V)_G = E(U,V)_G,$$

(1.3) $$\mathrm{Ric}_B(X,Y) - \frac{1}{2}\sum_i(\eta(X,X_i),\eta(Y,X_i))_G = E(X,Y)_B,$$

where $\{X_i\}$ is an orthonormal basis of tangent vectors at an arbitrary point $p$ in $B$, $X, Y$ are tangent vectors at $p$, and $U, V \in \mathfrak{t}$. Even these equations for $r = 1$ seem difficult to solve in general. So we will specialize further to a class of torus bundles over products of Kähler-Einstein manifolds with positive first Chern class. We consider for simplicity the case of circle bundles first.

Let $(M, g)$ be a Kähler-Einstein manifold with positive scalar curvature, or, equivalently, $c_1(M) > 0$. A theorem of S. Kobayashi [16] says that $M$ is simply connected, and hence $H^2(M;\mathbb{Z})$ has no torsion. We can therefore write $c_1(M) = q\alpha$, where $\alpha \in H^2(M;\mathbb{Z})$ is indivisible, and $q$ is a positive integer. Let $\omega(X,Y) = g(JX,Y)$ be the Kähler form, and $\rho(X,Y) = \mathrm{Ric}_g(JX,Y)$ the Ricci form. Then $\rho$ is harmonic and $[\rho] = 2\pi c_1(M)$. In the following we will normalize the metric $g$ on $M$ so that $[\omega] = 2\pi\alpha$. Since $\mathrm{Ric}_g = E\cdot g$ implies $\rho = E\cdot\omega$, this is equivalent to choosing $E = q$.

Now let $(M_i, g_i)$, $i = 1, \cdots, m$, be Kähler-Einstein manifolds with $c_1(M_i) > 0$ and real dimension $n_i$. Write $c_1(M_i) = q_i\cdot\alpha_i$ with $\alpha_i$ indivisible, and normalize $g_i$ such that $[\omega_i] = 2\pi\alpha_i$, or, equivalently, $\mathrm{Ric}_{g_i} = q_i\cdot g_i$. We consider principal circle bundles $P$ over $B = M_1\times\cdots\times M_m$. They are classified by their Euler class $e(P)$ in $H^2(B;\mathbb{Z})$. Let $\pi_i\colon B\to M_i$ denote the projection onto the $i$th factor.

**(1.4) Theorem.** *Let $(M_i, g_i)$, $i = 1, \cdots, m$, be Kähler-Einstein manifolds with $c_1(M_i) > 0$ as above, and $\pi\colon P\to B = M_1\times\cdots\times M_m$ be a principal circle bundle whose Euler class is $e(P) = \sum b_i\pi_i^*\alpha_i$, $b_i\in\mathbb{Z}$. Then, if $e(P)\neq 0$, $P$ carries an Einstein metric with positive scalar curvature* *uniquely characterized up to homothety by the requirements that $\pi$ is a Riemannian submersion with totally geodesic fibers and that the metric on $B$ is of the form $x_1 g_1 \perp \cdots \perp x_m g_m$ for some choice of $x_1, \cdots, x_m$.*

*Proof.* For the metric on $B$ we choose a product metric $( \, , \, )_B = x_1 g_m \perp \cdots \perp x_m g_m$, where $x_1, \cdots, x_m$ are positive constants to be determined later. The connection $\theta$ on $P$ has to be chosen such that $d\theta = \pi^* \eta$ with $\eta$ harmonic. Since $\omega_i$ is the unique harmonic representative in $2\pi\alpha_i$ with respect to $x_i g_i$, and since $[\eta] = 2\pi e(P)$, we need to choose $\eta = \sum b_i \pi_i^* \omega_i$. This uniquely determines $\theta$ up to gauge equivalence. Indeed, if $\theta'$ is any connection, then $d\theta' = \pi^* \eta'$ with $[\eta'] = [\eta]$, and hence $\eta' - \eta = d\alpha$. But then $\theta = \theta' - \pi^*(\alpha)$ satisfies $d\theta = \pi^* \eta$. Next, if $\theta$, $\theta'$ are two connections with $d\theta = d\theta' = \pi^* \eta$, then $\theta - \theta' = \pi^* \beta$ for some closed 1-form $\beta$ on $B$. In our case $H^1(B; \mathbb{R}) = 0$, and hence $\beta = df$. Then the gauge transformation $G_f \colon P \to P$ given by $x \mapsto x e^{if(\pi(x))}$ satisfies $G_f^* \theta = \theta'$.

Finally we choose the metric on the fiber such that it becomes a circle of length $2\pi\rho$. Then (1.2) becomes $2\rho^2 \|\eta\|^2 = 4E$. Since $\omega_i(X, Y) = g_i(JX, Y)$, we have $2\|\omega_i\|^2_{g_i} = n_i$, and hence $2\|\omega_i\|^2_{x_i g_i} = n_i / x_i^2$. Since $\eta = \sum_i b_i \pi_i^* \omega_i$, we have

$$\rho^2 \sum n_i \frac{b_i^2}{x_i^2} = 4E. \tag{1.5}$$

Similarly, to evaluate (1.3), we choose $X, Y$ tangent to $M_i$, and using $\mathrm{Ric}_{M_i} = q_i \cdot g_i$ we obtain

$$\frac{q_j}{x_j} - \frac{1}{2}\rho^2 \left(\frac{b_j}{x_j}\right)^2 = E, \qquad j = 1, \cdots, m. \tag{1.6}$$

We may assume that all the $b_i$ are nonzero; for otherwise, after permuting the $M_i$, the total space $P$ actually splits as $P' \times M_{k+1} \times \cdots \times M_m$, where $P'$ is a principal circle bundle over $M_1 \times \cdots \times M_k$ with Euler class $\sum_{i=1}^k b_i \pi_i^* \alpha_i$ and $b_i \neq 0$, $1 \leq i \leq k$. If an Einstein metric can be constructed on $P'$, then clearly one can be constructed on $P$.

We now substitute $\rho^2$ from (1.5) into (1.6) and obtain

$$\left(\frac{q_j}{x_j}\right) - E = 2E \left(\frac{b_j}{x_j}\right)^2 \left(\sum_i n_i \frac{b_i^2}{x_i^2}\right)^{-1}.$$

With the new variable $s_i = q_i / (x_i E)$, this becomes

$$(s_j - 1) \sum_{i=1}^m n_i \left(\frac{b_i}{q_i}\right)^2 s_i^2 = 2 \left(\frac{b_j}{q_j}\right)^2 s_j^2, \qquad j = 1, \cdots, m, \tag{1.7}$$

or, equivalently,

$$\left(1+\frac{2}{n_j}-s_j\right)\sum_i n_i\left(\frac{b_i}{q_i}\right)^2 s_i^2=\frac{2}{n_j}\sum_{i\neq j} n_i\left(\frac{b_i}{q_i}\right)^2 s_i^2.$$

These equations imply in particular that

$$(1.8)\qquad 1<s_j<1+\frac{2}{n_j}\quad\text{or}\quad \frac{q_j}{E}\left(\frac{n_j}{n_j+2}\right)<x_j<\frac{q_j}{E}.$$

If we multiply each equation in (1.7) by $n_j$ and add, we get $\sum n_j(s_j-1)=2$. Hence (1.7) is equivalent to the system

$$(1.9a)\qquad (b_j/q_j)^2 s_j^2/(s_j-1)=c\quad\text{for some } c>0 \text{ for all } j,$$

$$(1.9b)\qquad \sum n_j(s_j-1)=2.$$

Observe that on $1\leq s\leq 2$, $s^2/(s-1)$ is monotone and decreases form $\infty$ to 4. Hence for each $c$ with $c(q_j/b_j)^2>4$ for all $j$, there exists a unique solution of (1.9a) with $1<s_j<2$. If $c$ is very large, all $s_j$ will be close to 1 and so $\sum n_j(s_j-1)<2$. As $c$ decreases, all $s_j$ increase monotonically. For some value of $c$, the largest component $s_j$ will be equal to 2, at which point $\sum n_j(s_j-1)>2$ since $n_j\geq 2$. Therefore, there exists a unique value of $c$ with $\sum n_j(s_j-1)=2$, and hence a unique solution $s_1,\cdots,s_m$ of (1.9). q.e.d.

We now generalize (1.4) to principal $T^r$ bundles. A principal $T^r$ bundle $\pi\colon P\to B$ is classified by $r$ elements $\beta_1,\cdots,\beta_r\in H^2(B;\mathbb{Z})$. $T^r$ comes with a canonical decomposition $T^r=S^1\times\cdots\times S^1$, and hence the Lie algebra $\mathfrak{t}^r$ has a canonical basis $\{e_1,\cdots,e_r\}$. $\beta_i$ can be described as the Euler class of the orientable circle bundle $P/T^{r-1}\to B$ where $T^{r-1}\subset T^r$ is the subtorus with the $i$th $S^1$ factor deleted. If $A$ is an automorphism of $T^r$, we can change the action of $T^r$ on $P$ via this automorphism to get a new principal $T^r$ bundle. This automorphism gives rise to a new decomposition of $T^r$ into a product of circles, and hence to a new basis $e_i'=A_{ij}e_j$, where $A$ is an integral matrix with $\det A=1$. This new $T^r$ bundle has characteristic classes $\beta_1',\cdots,\beta_r'$, where $\beta_i'=(A^T)^{-1}_{ij}\beta_j$.

If $\theta\colon P\to\mathfrak{t}^r$ is a principal $T^r$ connection with curvature $\Omega=d\theta=\pi^*\eta$, we can write $\eta=\sum\eta_i e_i$, and then $[\eta_i]=2\pi\beta_i$. In the spectral sequence for the bundle $T^r\to P\to B$, the generators in $H^1(T^r;\mathbb{Z})$ transgress to $\beta_i$, and hence $\pi_1(P)$ is finite iff $H^1(P;\mathbb{Z})=0$, which holds iff $H^1(B;\mathbb{Z})=0$ and $\beta_1,\cdots,\beta_r$ are linearly independent in $H^2(B;\mathbb{Z})$ (in particular they are not torsion classes). Since we want to construct Einstein metrics on $P$ with $E>0$, we need $\pi_1(P)$ finite by Bonnet-Myers.

**(1.10) Theorem.** *Let $(M_i, g_i)$, $1 \leq i \leq m$, be Kähler-Einstein manifolds with $c_1(M_i) > 0$, and $\pi: P \to B = M_1 \times \cdots \times M_m$ be the principal $T^r$ bundle, $r \leq m$, with characteristic classes $\beta_i = \sum_{j=1}^m b_{ij}\pi_j^*\alpha_j$, $i = 1, \cdots, r$, where $b_{ij} \in \mathbb{Z}$. Then if $B = (b_{ij})$ has maximal rank, there exists an Einstein metric on $P$ with positive scalar curvature such that $\pi$ is a Riemannian submersion with totally geodesic flat fibers and such that the metric on the base is a product of the Kähler-Einstein metrics.*

*Proof.* We again normalize the metric $g_i$ on $M_i$ such that $[\omega_i] = 2\pi\alpha_i$ and choose $(\ ,\ )_B = x_1 g_1 \perp \cdots \perp x_m g_m$. If $\sigma_i: P_i \to B$ is the principal circle bundle with $e(P_i) = \beta_i$, then there exists a principal connection $\theta_i$ on $P_i$, unique up to gauge equivalence, such that $d\theta_i = \pi^*(\sum_{j=1}^m b_{ij}\omega_j)$. $\pi: P \to B$ is then the pull-back bundle via the diagonal map $\Delta$:

$$\begin{array}{ccc} P & \longrightarrow & P_1 \times \cdots \times P_m \\ \downarrow \pi & & \downarrow \sigma_1 \times \cdots \times \sigma_r \\ B & \stackrel{\Delta}{\longrightarrow} & B \times \cdots \times B \end{array}$$

We let $\theta$ be the principal connection on $P$ given by the pullback of $\theta_1 \times \cdots \times \theta_r$. Then $d\theta = \Omega = \pi^*\eta$ with $\eta = \sum_{k=1}^r \eta_k e_k$ and $\eta_k = \sum_{j=1}^m b_{kj}\omega_j$. The 2-form $\eta$ is harmonic with respect to the metric $(\ ,\ )_B$. We now choose an arbitrary left-invariant metric on the torus $T^r$, given by a symmetric matrix $h_{kl} = (e_k, e_l)_{T^r}$. As before, (1.2) and (1.3) become

$$\sum_{i=1}^m \frac{b_{ki}b_{li}n_i}{x_i^2} = 4Eh^{kl}, \qquad 1 \leq k, l \leq r, \tag{1.11}$$

$$\frac{q_i}{x_i} - \frac{1}{2}\sum_{l=1}^r \frac{h_{kl}b_{ki}b_{li}}{x_i^2} = E, \qquad i = 1, \cdots, m. \tag{1.12}$$

The first equation determines the metric on the torus in terms of the $x_i$, which we will substitute into (1.12) to obtain a system of $m$ equations for the $x_i$.

We also introduce the auxiliary Euclidean vector space $\mathbb{R}^m$ with orthonormal basis $\{f_1, \cdots, f_m\}$. Define the vectors $v_1, \cdots, v_r \in \mathbb{R}^m$ by $v_k = \sum_j b_{ij}(\sqrt{n_j}/x_j)f_j$ and the matrix $Q_{kl} = \langle v_k, v_l\rangle$. Since the matrix $B$ has maximal rank, the $v_i$ are linearly independent, and hence $\det Q \neq 0$. (1.11) now reads $Q_{kl} = 4Eh^{kl}$ and hence $h_{kl} = 4EQ^{kl}$. For each multi-index $J$, $1 \leq j_1 \leq \cdots \leq j_r \leq m$, we let $f_J = f_{j_1} \wedge \cdots \wedge f_{j_r}$ and then

$$\det Q = \langle v_1 \wedge \cdots \wedge v_r, v_1 \wedge \cdots \wedge v_r\rangle = \sum_J \langle v_1 \wedge \cdots \wedge v_r, f_J\rangle^2. \tag{1.13}$$

Similarly,

$$
\begin{aligned}
h_{kl} &= 4EQ^{kl} \\
&= 4E(\det Q)^{-1}(-1)^{k+l}\langle v_1 \wedge \cdots \wedge \check{v}_l \wedge \cdots \wedge v_r, \\
&\qquad\qquad v_1 \wedge \cdots \wedge \check{v}_k \wedge \cdots \wedge v_r\rangle \\
&= 4E(\det Q)^{-1}(-1)^{k+l}\sum_I [\langle v_1 \wedge \cdots \wedge \check{v}_l \wedge \cdots \wedge v_r, f_I\rangle \\
&\qquad\qquad \cdots \langle v_1 \wedge \cdots \wedge \check{v}_k \wedge \cdots \wedge v_r, f_I\rangle],
\end{aligned} \tag{1.14}
$$

where $I$ runs over all ordered multi-indices of length $r-1$. Thus we obtain

$$
\begin{aligned}
\sum_{k,l} \frac{h_{kl}b_{ki}b_{li}}{x_i^2} &= \frac{4E}{n_i}\sum_{k,l} Q^{kl}\langle v_k, f_i\rangle\langle v_l, f_i\rangle \\
&= \frac{4E}{n_i \det Q}\sum_I \sum_{k,l}(-1)^k\langle v_k, f_i\rangle\langle v_1 \wedge \cdots \wedge \check{v}_k \wedge \cdots \wedge v_r, f_I\rangle(-1)^l \\
&\qquad \cdot \langle v_l, f_i\rangle\langle v_1 \wedge \cdots \wedge \check{v}_l \wedge \cdots \wedge v_r, f_I\rangle \\
&= \frac{4E}{n_i \det Q}\sum_I \left[\sum_k (-1)^k\langle v_k, f_i\rangle\langle v_1 \wedge \cdots \wedge \check{v}_k \wedge \cdots \wedge v_r, f_I\rangle\right]^2 \\
&= \frac{4E}{n_i \det Q}\sum_{J, i\in J}\langle v_1 \wedge \cdots \wedge v_r, f_J\rangle^2,
\end{aligned}
$$

since the expression in [ ] is up to a sign the Laplace development of $\langle v_1 \wedge \cdots \wedge v_r, f_J\rangle$, $J = I \cup \{i\}$, according to column $i$, and hence is 0 if $i \in I$.

We now introduce as before the new variable $s_j = q_j/(x_j E)$ and define

$$
a_J = \langle \overline{v}_1 \wedge \cdots \wedge \overline{v}_r, f_J\rangle^2, \quad \text{where } \overline{v}_k = \sum_{j=1}^m b_{kj}(\sqrt{n_j}E/q_j)f_j.
$$

Then $\langle v_1 \wedge \cdots \wedge v_r, f_J\rangle^2 = a_J s_J^2$ where $s_J^2 = s_{j_1}^2 \cdots s_{j_r}^2$. Hence (1.12) becomes

$$
E(s_i - 1) = \frac{2E}{n_i \det Q}\sum_{J, i\in J} a_J s_J^2,
$$

or

$$
n_i(s_i - 1)\sum_J a_J s_J^2 = 2\sum_{J, i\in J} a_J s_J^2, \qquad i = 1, \cdots, m, \tag{1.15}
$$

or equivalently,

$$
(2 - n_i(s_i - 1))\sum_J a_J s_J^2 = 2\sum_{J, i\notin J} a_J s_J^2.
$$

If we write $J, i \in J$ as $I$, $i$ where $J = I \cup (i)$, then we get

$$\sum_{J, i \in J} a_J s_J^2 = \left( \sum_{I, i \notin I} a_{I,i} s_I^2 \right) s_i^2 .$$

We can assume that for each $i$ there exists a multi-index $I$ with $i \notin I$ and $a_{I,i} > 0$, since otherwise (1.15) implies $s_i = 1$ and we can reduce the system of equations. Hence we get

$$(1.16) \qquad 1 < s_i < 1 + \frac{2}{n_i} \quad \text{or} \quad \frac{n_i q_i}{(n_i + 2)E} < x_i < \frac{q_i}{E},$$

and $\sum n_i(s_i - 1) = 2r$. It follows that (1.15) is equivalent to

$$(1.17a) \qquad \frac{s_i^2}{n_i(s_i - 1)} \sum_{I, i \notin I} a_{I,i} s_I^2 = c \quad \text{for some } c > 0,\ i = 1, \cdots, m,$$

$$(1.17b) \qquad \sum n_i(s_i - 1) = 2r.$$

Let $\psi_i(s) = \{s_i^2/[n_i(s_i - 1)]\} \sum_{I, i \notin I} a_{I,i} s_I^2$. Then $\psi = (\psi_1, \cdots, \psi_m)$ defines a continuous map of $A = \{(s_1, \cdots, s_m) | s_i > 1\}$ to $B = \{(x_1, \cdots, x_m | x_i > 0\}$. In $A$ we define the open simplex $\Delta = \{s \in A | \sum n_i(s_i - 1) = 2r\}$ and the closed simplices $\Delta_\varepsilon = \{(s_1, \cdots, s_m) | \sum n_i(s_i - 1) = 2r,\ s_i \geq 1 + \varepsilon\}$ for $\varepsilon > 0$. We wish to show that $\psi(A)$ intersects the diagonal $D = \{(x, \cdots, x) | x > 0\} \subset B$. For this purpose we will show that for each sufficiently small $\varepsilon > 0$, $\psi \colon \partial\Delta_\varepsilon \to B \setminus D \simeq S^{m-2}$ has nonzero degree. Indeed, it is clear that for $\varepsilon$ sufficiently small, $\psi(\partial\Delta_\varepsilon) \cap D = \emptyset$ and to see that $\psi | \partial\Delta_\varepsilon$ has nonzero degree, we define the homotopy $\psi^t$:

$$\psi_i^t = [(1 - t)s_i^2 \sum_{I, i \notin I} a_{I,i} s_I^2 + t n_i]/(s_i - 1)n_i$$

with $\psi^0 = \psi$ and $\psi_i^1 = 1/(s_i - 1)$. For sufficiently small $\varepsilon > 0$, $\psi^t(\partial\Delta_\varepsilon) \cap D = \emptyset$ for all $t$, $0 \leq t \leq 1$. Since $\psi^1$ is a homeomorphism of $A$ onto $B$, it follows that $\psi \colon \partial\Delta_\varepsilon \to B \setminus D$ has degree $\pm 1$. Hence (1.17) has at least one solution $(s_1, \cdots, s_m)$.

**Remark.** Notice that if two principal torus bundles differ by an automorphism of $T^r$, the corresponding Einstein metrics we constructed are isometric to each other.

## 2. Topological properties and group actions

In this section we examine some of the topological and symmetry properties of our Einstein manifolds.

If $m = 1$, it follows easily that the total space of the circle bundle $P$ is simply connected iff $c_1(P) = \alpha$. The circle bundle $P$ with $c_1(P) = k\alpha$ is just $P/\mathbb{Z}_k$ where $\mathbb{Z}_k \subset S^1$.

If $m = 2$ and $c_1(P) = k\alpha_1 + l\alpha_2$, then $P$ is simply connected iff $k$ and $l$ are relatively prime. Assuming this, one can also describe $P$ as the base space of another circle bundle as follows. Let $P_i$ be the total space of the circle bundle over $M_i$ with $c_1(P_i) = \alpha_i$. $P_i$ is simply connected and $P_1 \times P_2$ is a $T^2$ bundle over $M_1 \times M_2$. On $P_1 \times P_2$ we have the $S^1$ action given by $e^{i\theta}(x, y) = (e^{il\theta}x, e^{-ik\theta}y)$. Since $k$ and $l$ are relatively prime, this $S^1$ action is free, and the quotient $(P_1 \times P_2)/S^1$, which is necessarily simply connected, becomes a $T^2/S^1 \simeq S^1$ bundle over $M_1 \times M_2 = (P_1 \times P_2)/T^2$. The Euler class is easily seen to be $\pm(k\alpha_1 + l\alpha_2)$, the sign depending on the isomorphism $S^1 \approx T^2/S^1$.

More generally, one can describe the simply connected principal $T^r$ bundles over $M_r \times \cdots \times M_m$ in a similar fashion. Let $P_1 \times \cdots \times P_m$ be the principal $T^m$ bundle over $M_1 \times \cdots \times M_m$ and $T^{m-r} \subset T^m$ be a compact subtorus. Then $(P_1 \times \cdots \times P_m)/T^{m-r}$ is a $T^m/T^{m-r} \approx T^r$-bundle. The actual structure as a principal $T^r$-bundle depends on the isomorphism chosen, and two different isomorphisms give rise to principal bundles which differ by an automorphism of $T^r$ (i.e., an element of $\mathrm{SL}(r, \mathbb{Z})$ if a basis of $\mathfrak{t}^r$ is chosen as in §1).

We specialize now to the simplest examples with $m = 2$. Let $M_1 \times M_2 = P^p\mathbb{C} \times P^q\mathbb{C}$; then $\alpha_i$ generates $H^2(M_i; \mathbb{Z})$, so every principal circle bundle over $M_1 \times M_2$ has Euler class of the form $k\alpha_1 + l\alpha_2$ and hence is one of our examples. We denote the corresponding total spaces by $M_{k,l}^{p,q}$. Of course, in the notation of the previous paragraphs, $P_1 = S^{2p+1}$ and $P_2 = S^{2q+1}$, so $M_{k,l}^{p,q} = (S^{2p+1} \times S^{2q+1})/S^1$ if $(k, l) = 1$, where $S^1$ acts via $e^{i\theta}(x, y) = (e^{il\theta}x, e^{-ik\theta}y)$. Clearly, $\mathrm{SU}(p+1) \times \mathrm{SU}(q+1)$ acts transitively on $M_{k,l}^{p,q}$.

From now on we will assume that $(k, l) = 1$ and $1 \leq p \leq q$. Since for $k = 0$ (resp. $l = 0$) we get $P^p\mathbb{C} \times S^{2q+1}$ (resp. $S^{2p+1} \times P^q\mathbb{C}$) we will also assume that $k, l \neq 0$. In the following we examine the topological properties of these spaces, beginning with their cohomology rings and characteristic classes.

**(2.1) Proposition.**

(a) $H^*(M_{k,l}^{p,q}; \mathbb{Z}) = \mathbb{Z}[x_2, y_{2q+1}]/((lx)^{p+1}, x^{q+1}, x^{p+1}y, y^2)$.

(b) *The total Stiefel-Whitney class is* $w(M_{k,l}^{p,q}) = (1 + lx)^{p+1}(1 + kx)^{q+1}$ *and the total Pontrjagin class is* $p(M_{k,l}^{p,q}) = (1 + l^2x^2)^{p+1}(1 + k^2x^2)^{q+1}$. *So,* *in particular*

$$\omega_2(M^{p,q}_{k,l}) = (l(p+1) + k(q+1))x \pmod 2,$$

$$p_1(M^{p,q}_{k,l}) = [(p+1)l^2 + (q+1)k^2]x^2,$$

$$p_2(M^{p,q}_{k,l}) = \left[\binom{p+1}{2}l^4 + (p+1)(q+1)k^2l^2 + \binom{q+1}{2}k^4\right]x^4.$$

*Proof.* One can derive this from the Gysin sequence of the $S^1$ bundle, but a simpler argument is as follows.

One has the commutative diagram of fibrations

$$\begin{array}{ccccc} S^{2p+1}\times S^{2q+1} & \xrightarrow{\sigma_1} & M^{p,q}_{k,l} & \xrightarrow{\pi_i} & BS^1 \\ \downarrow \mathrm{id} & & \downarrow \tau_1 & & \downarrow \tau_2 \\ S^{2q+1}\times S^{2q+1} & \xrightarrow{\sigma_2} & P^p\mathbb{C}\times P^q\mathbb{C} & \xrightarrow{\pi_2} & BS^1\times BS^1 \end{array}$$

coming from the $S^1$ bundle $\sigma_1$ and the $S^1 \times S^1$ bundle $\sigma_2$. The inclusion $S^1 \to S^1 \times S^1$ is given by $e^{i\theta} \mapsto (e^{il\theta}, e^{-ik\theta})$ and induces the map $\tau_2\colon BS^1 \to BS^1 \times BS^1$. If we let $H^*(BS^1;\mathbb{Z}) = \mathbb{Z}[s]$ and $H^*(BS^1 \times BS^1, \mathbb{Z}) = \mathbb{Z}[t_1, t_2]$, then $\tau_2^*(t_1) = ls$ and $\tau_2^*(t_2) = -ks$. $\sigma_2$ is just the product of the two circle bundles $S^{2p+1} \to P^p\mathbb{C}$ and $S^{2q+1} \to P^q\mathbb{C}$. $\tau_1$ is the bundle defining $M^{p,q}_{k,l}$. If we set $H^*(S^{2p+1} \times S^{2q+1};\mathbb{Z}) = \Lambda(u, v)$, it follows that the only nonzero differentials in the spectral sequence of $\pi_2$ are given by $d_{2p+2}(u) = t_1^{p+1}$ and $d_{2q+2}(v) = t_2^{q+1}$. By naturality, the differentials in the spectral sequence of $\pi_1$ are given by $d_{2p+2}(u) = (ls)^{p+1}$ and $d_{2q+2}(v) = (-ks)^{q+1}$. From this the cohomology ring structure follows easily.

To prove (b), we observe that the above argument also implies that $\tau_1^*$ sends the generator in $H^2(P^p\mathbb{C};\mathbb{Z})$ to $lx$ and the generator in $H^2(P^q\mathbb{C};\mathbb{Z})$ to $(-k)x$. Furthermore, the tangent bundle of $M^{p,q}_{k,l}$ splits into the direct sum of the bundle along the fibers and the pull-back under $\tau_1$ of the tangent bundle of $P^p\mathbb{C} \times P^q\mathbb{C}$. The vertical bundle is trivial, a trivialization given by the $S^1$ action. Hence characteristic classes of $M^{p,q}_{k,l}$ are the pull-back under $\tau_1^*$ of the characteristic classes of $P^p\mathbb{C} \times P^q\mathbb{C}$. This implies (b).

In particular we have

$$H^*(M^{p,q}_{k,l};\mathbb{Q}) = H^*(P^p\mathbb{C} \times S^{2q+1};\mathbb{Q}),$$

$$H^*(M^{p,p}_{k,l};\mathbb{Z}) = H^*(P^p\mathbb{C} \times S^{2p+1};\mathbb{Z}),$$

$$H^*(M^{p,q}_{k,\pm1};\mathbb{Z}) = H^*(P^p\mathbb{C} \times S^{2q+1};\mathbb{Z}),$$

but in all other cases the cohomology has some torsion.

The Pontrjagin classes are by definition diffeomorphism invariants and by Novikov the rational Pontrjagin classes are homeomorphism invariants.

Notice that $p_1$ is rationally nonzero iff $p \geq 2$ and $p_2$ iff $p \geq 4$. Hence we obtain

**(2.2) Corollary.** (a) *For* $2 \leq p < q$, $M_{k,l}^{p,q}$ *is homeomorphic* (*or diffeomorphic*) *to* $M_{k',l'}^{p,q}$ *iff* $|k| = |k'|$ *and* $|l| = |l'|$.

(b) *For* $p \geq 2$ *there are infinitely many homeomorphism types among the manifolds* $M_{k,l}^{p,p}$, *which all have the same integral cohomology ring. If* $p \geq 4$, *then* $M_{k,l}^{p,p}$ *is homeomorphic* (*or diffeomorphic*) *to* $M_{k',l'}^{p,p}$ *iff* $(|k|, |l|)$ *is a permutation of* $(|k'|, |l'|)$.

Hence we obtain examples of infinitely many homotopy types of Einstein manifolds in every odd dimension $\geq 7$ and also examples of infinitely many Einstein manifolds with the same integral cohomology ring but different homeomorphism types.

In contrast to the above results, we find the following different behavior when $p = 1$.

**(2.3) Proposition.** (a) $M_{k,l}^{1,1}$ *are all diffeomorphic to* $S^3 \times S^2$.

(b) $M_{k,\pm 1}^{1,q}$ *is diffeomorphic to* $S^2 \times S^{2q+1}$ *if* $q$ *is odd or if* $q$ *is even and* $k$ *is even. If* $q$ *is even and* $k$ *is odd*, $M_{k,\pm 1}^{1,q}$ *is diffeomorphic to the total space of the unique nontrivial* $S^{2q+1}$ *bundle over* $S^2$.

(c) $M_{k,\pm 2}^{1,q}$ *is diffeomorphic to a nontrivial* $P^{2q+1}\mathbb{R}$ *bundle over* $S^2$, *which for* $q$ *odd is independent of* $k$ *and for* $q$ *even depends on the congruence class of* $k \bmod 4$.

(d) *If we fix* $q$ *and* $|l|$, *there are only finitely many diffeomorphism types among the manifolds* $M_{k,l}^{1,q}$.

*Proof.* (a) $M_{k,l}^{1,1}$ is a compact simply connected five-dimensional spin manifold, and hence we can apply the classification for such manifolds by Smale [26]. They are determined up to diffeomorphism by $H^2(M; \mathbb{Z})$. By (2.1), $H^2(M_{k,l}^{1,1}) = \mathbb{Z}$ and hence they are all diffeomorphic to $S^2 \times S^3$. Surprisingly, it seems to be difficult to find an explicit diffeomorphism.

(b), (c). The $S^1$ action on $S^{2p+1} \times S^{2q+1}$ can be divided out in two stages. First we divide by $\mathbb{Z}_l \subset S^1$ to get $S^{2p+1} \times L_{|l|}^{2q+1}$, where $L_{|l|}^{2q+1}$ is the associated lens space. The residual circle action is given by $e^{i\theta}(u, [v]) = (e^{i\theta}u, [e^{-i(k/l)\theta}v])$. Hence $M_{k,l}^{p,q}$ has been written as the associated fiber bundle to the standard Hopf bundle $S^{2p+1} \to P^p\mathbb{C}$ with fiber $L_{|l|}^{2q+1}$ having circle action $e^{i\theta}[v] = [e^{i(k/l)\theta}v]$.

If $p = 1$ and $|l| = 1$, we get an $S^{2q+1}$ bundle over $S^2$ whose classifying map is given by $f\colon S^1 \to \mathrm{SO}(2q+2)$, $f(e^{i\theta}) = \mathrm{diag}(R(\pm k\theta), \cdots, R(\pm k\theta))$, where $R(\theta)$ is a 2-dimensional rotation through angle $\theta$. It follows that $f$ represents the element $(q+1)k \pmod 2$ in $\pi_1(\mathrm{SO}(2q+2)) = \mathbb{Z}_2$, which proves (b) since by [12] there are precisely two homotopy types among the total spaces of these sphere bundles.

If $p = 1$ and $|l| = 2$, we get a $P^{2q+1}\mathbb{R}$ bundle over $S^2$. The structural group of this bundle is $SO(2q+2)/\pm\mathrm{Id}$, and the classifying map is $f\colon e^{i\theta} \mapsto \mathrm{diag}(R(\frac{1}{2}k\theta), \cdots, R(\frac{1}{2}k\theta))/\pm\mathrm{Id}$. Recall that $\pi_1(SO(2q+2)/\pm\mathrm{Id}) = \mathbb{Z}_4$ if $q$ is even and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if $q$ is odd. Since $(k, l) = 1$, $k$ must be odd and hence $f$ is nonzero in $\pi_1$. To determine whether $[f]$ depends on $k$ or not, we lift $f$ to a path in $\mathrm{Spin}(2q+2)$ from Id to an element of $Z(\mathrm{Spin}(2q+2)) \simeq \pi_1(SO(2q+2)/\pm\mathrm{Id})$. One easily shows that the rotation $R(\frac{1}{2}k\theta)$ in the $e_1 - e_2$ plane lifts to

$$(\cos(\tfrac{1}{8}k\theta)e_1 + \sin(\tfrac{1}{8}k\theta)e_2)(-\cos(\tfrac{1}{8}k\theta)e_1 + \sin(\tfrac{1}{8}k\theta)e_2) = \cos(\tfrac{1}{4}k\theta) + \sin(\tfrac{1}{4}k\theta)e_1e_2.$$

Hence for $\theta = 2\pi$ we get $i^{k-1}e_1e_2$. It follows that the lift of $f$ ends at $i^{(k-1)(q+1)}\omega$, where $\omega$ is the volume element in $\mathrm{Spin}(2q+2)$. This proves (c).

(d) To prove (d), we apply the classification results of Sullivan (see [27, Theorem 13.1 and the proof of Theorem 12.5]). They imply that if a collection of simply connected closed manifolds of dimension $\geq 5$ all have isomorphic integral cohomology rings, the same rational Pontrjagin classes, and if their minimal model is a formal consequence of their rational cohomology ring, then there are only finitely many diffeomorphism types among them. For more details, especially about the role of the integral lattices Sullivan defined, see [19], Proposition 2.1]. Hence (d) follows from (2.1), because $H^*(M_{k,l}^{1,q}; Q) = H^*(S^2 \times S^{2q+1}; Q)$ implies that the minimal model is clearly formal. q.e.d.

Analogous calculations can easily be made for the principal torus bundles over $P^{k_1}\mathbb{C} \times P^{k_2}\mathbb{C} \times \cdots \times P^{k_m}\mathbb{C}$ with similar results. We obtain in particular even-dimensional examples with analogous properties as the above odd-dimensional ones. We mention specifically below only the following results:

The $T^2$ bundles over $P^p\mathbb{C} \times P^q\mathbb{C} \times P^r\mathbb{C}$ with simply connected total space can be described as $M_{k,l,s}^{p,q,r} = (S^{2p+1} \times S^{2q+1} \times S^{2r+1})/S^1$, where $S^1$ acts via $e^{i\theta}(x, y, z) = (e^{ik\theta}x, e^{il\theta}y, e^{is\theta}z)$ and $\gcd(k, l, s) = 1$.

All $T^2$ bundles over $P^1\mathbb{C} \times P^1\mathbb{C} \times P^1\mathbb{C}$ have the same integral cohomology ring and vanishing Pontrjagin classes. They are most likely all diffeomorphic to $S^3 \times S^3 \times S^2$.

If $p = q = r > 1$, the integral cohomology ring is independent of $k$, $l$, $s$, but the Pontrjagin classes imply that most of them are not homeomorphic to each other and not homeomorphic to $M_{k',l'}^{p,q} \times S^{2p+1}$.

If $p < q < r$, the cohomology ring depends only on $k$, $d = \gcd(k^{p+1}, l^{q+1})$ and $k^{p+1}/d$. Hence we obtain infinitely many homotopy types in all even dimensions $\geq 14$. If $p = 1$, Sullivan's theorem again shows that if we fix $k$ and $d$ then there are only finitely many diffeomorphism types among the $M^{1,q,r}_{k,l,s}$.

The remainder of this section is devoted to studying the homogeneous structures on $M^{p,q}_{k,l}$.

The spaces $M^{p,q}_{k,l}$ can be written as the homogeneous spaces

$$\frac{\mathrm{U}(p+1) \times \mathrm{U}(q+1)}{\mathrm{U}(p) \times \mathrm{U}(q) \times \mathrm{U}_{kl}},$$

where $\mathrm{U}_{kl}$ is the subgroup

$$\left\{ \begin{pmatrix} I_p & 0 \\ 0 & e^{il\theta} \end{pmatrix}, \begin{pmatrix} I_q & 0 \\ 0 & e^{-ik\theta} \end{pmatrix} \right\},$$

and the isotropy representation is given by $1 \oplus [\mu_p \otimes 1 \otimes \Phi^{-l}]_{\mathbb{R}} \oplus [1 \otimes \mu_q \otimes \Phi^{k}]_{\mathbb{R}}$, where $\Phi$ is the usual one-dimensional representation of the circle. In this form, the ineffective kernel is a circle $Z_{kl}$. The semisimple subgroup $\mathrm{SU}(p+1) \times \mathrm{SU}(q+1)$ also acts transitively on $M^{p,q}_{k,l}$ with finite ineffective kernel. When infinitely many of the $M^{p,q}_{k,l}$ are diffeomorphic or homeomorphic, then it is interesting to compare the transitive group actions. For example, some of the effective semisimple transitive groups $G_{k,l}$ are listed below in Table 1. For each fixed smooth (resp. topological) manifold $M$ and effective abstract group $G$, we ask when the various transitive actions are equivalent group actions and when they are conjugate as subgroups of the diffeomorphism (resp. homeomorphism) group.

**Table 1**

| | underlying smooth manifold $M$ | effective transitive abstract semisimple group $G$ |
|---|---|---|
| A(i) | $S^2 \times S^{2q+1} \cong M^{1,q}_{k,\pm 1}$, $k$ even, $q > 1$ | $\mathrm{SO}(3) \times \mathrm{SU}(q+1)$ |
| A(ii) | $S^2 \times S^{2q+1} \cong M^{1,q}_{k,\pm 1}$, $k$ odd, $q$ odd $> 1$ | $(\mathrm{SU}(2) \times \mathrm{SU}(q+1))/\Delta(\mathbb{Z}_2)$ |
| B | $S^2 \times_T S^{2q+1} \cong M^{1,q}_{k,\pm 1}$, $k$ odd, $q$ even | $\mathrm{SU}(2) \times \mathrm{SU}(q+1)$ |
| C(i) | $S^2 \times S^3 \cong M^{1,1}_{k,l}$, $k, l$ odd | $\mathrm{SO}(4)$ |
| C(ii) | $S^2 \times S^3 \cong M^{1,1}_{k,l}$, one of $k, l$ even | $\mathrm{SO}(3) \times \mathrm{SU}(2)$ |

**(2.4) Proposition.** (a) *If* $M_{k,l}^{1,q}$ *is diffeomorphic (or homeomorphic) to* $M_{k',l'}^{1,q}$, $q > 1$, *and if* $G_{k,l}$ *is isomorphic to* $G_{k',l'}$, *then the actions are equivalent iff* $|l| = |l'|$ *and* $k = k'$, *and as subgroups of* Diff$(M)$ (*resp.* Homeo$(M)$), *they are conjugate iff* $|k| = |k'|$ *and* $|l| = |l'|$.

(b) *In cases* C(i), *or* C(ii), *the actions of* $G$ *on* $M_{k,l}^{1,1}$ *and* $M_{k',l'}^{1,1}$ *are equivalent iff* $|k| = |k'|$ *and* $|l| = |l'|$, *and as subgroups of* Diff$(S^2 \times S^3)$ (*resp.* Homeo$(S^2 \times S^3)$) *they are conjugate iff* $(|k|, |l|)$ *is a permutation of* $(|k'|, |l'|)$.

*Proof.* The proposition follows easily from the following two facts. First, if $G$ acts transitively and effectively on $M$ in two ways then the two actions are equivalent iff for some point in $M$ the isotropy groups are conjugate in $G$. Second, if $i_1 \colon G \to \text{Diff}(M)$ and $i_2 \colon G \to \text{Diff}(M)$ are two embeddings, then $i_1(G)$ and $i_2(G)$ are conjugate in Diff$(M)$ iff the isotropy subgroups of some point in $M$ are conjugate by an automorphism of $G$. The automorphism is outer iff the transitive actions $i_1$ and $i_2$ are inequivalent. This can be seen as follows. Let $\Phi \in \text{Diff}(M)$ be such that $i_2(G) = \Phi i_1(G)\Phi^{-1}$. The association $g \mapsto g'$ given by $i_2(g') = \Phi i_1(g)\Phi^{-1}$ is an automorphism $\alpha$ of $G$. If $x_0 \in M$ and $H_i$ are the corresponding isotropy subgroups, then for $h \in H_1$, since $\Phi(x_0) = i_2(g_0)x_0$ for some $g_0 \in G$,

$$\Phi(i_1(h)x_0) = \Phi(x_0) = i_2(g_0)x_0 = i_2(\alpha(h))\Phi(x_0) = i_2(\alpha(h))i_2(g_0)x_0.$$

It follows that $\text{Ad}(g_0^{-1})\alpha(H_1) = H_2$. The converse is obvious and the result is the same if Diff$(M)$ is replaced by Homeo$(M)$. q.e.d.

As was remarked, the spaces $M_{k,l}^{1,q}$ have $G'_{k,l} = (\text{U}(2) \times \text{U}(q+1))/Z_{kl}$ as effective transitive groups, and $G_{k,l} \subsetneq G'_{k,l}$ is a maximal semisimple subgroup. So statements analogous to (2.4)(a),(b) hold for $G'_{k,l}$. In the following, we will show that except in special cases, $G'_{k,l}$ are not contained in any larger compact connected group which acts effectively and transitively on $M_{k,l}^{1,q}$, i.e., they are maximal compact connected subgroups of Diff$(M_{k,l}^{1,q})$ and Homeo$(M_{k,l}^{1,q})$.

**(2.5) Proposition.** *Suppose that* $(k, l) = 1$ *and* $kl \neq 0$. *Then the maximal compact connected Lie groups which can act transitively and effectively on* $M_{k,l}^{1,q}$ *are*

(a) $G'_{kl} = (\text{U}(2) \times \text{U}(q+1))/Z_{kl} = (\text{SU}(2) \times \text{SU}(q+1) \times S^1)/N_{kl}$ *for some finite normal subgroup* $N_{kl}$,

(b) $\text{SO}(3) \times \text{SO}(2q+2)$ *when* $l = \pm 1$.

*When* $l = \pm 1$, *the groups in* (a) *cannot be conjugate to a subgroup of* $\text{SO}(3) \times \text{SO}(2q+2)$ *in* Diff$(S^2 \times S^{2q+1})$ *or* Homeo$(S^2 \times S^{2q+1})$.

*Proof.* Let $M = M_{k,l}^{1,q}$ and $G'$ be a compact connected Lie group acting almost effectively and transitively on $M$ with isotropy group $H'$. $M$ is simply connected, $\pi_2 M \cong \mathbb{Z}$, and for $k \geq 3$, $\pi_k M \cong \pi_k(P^1\mathbb{C} \times P^q\mathbb{C}) \cong \pi_k S^3 \oplus \pi_k S^{2q+1}$. Recall that the rank of $M$, $r(M) = \sum_{k=1}^{\infty} \dim_Q[\pi_{2k+1}(M) \otimes Q]$, is two in this case. Then [23, Theorem 3, p. 159] says that there is a compact ideal $\mathfrak{g} \subset \mathfrak{g}'$ consisting of at most two simple ideals such that the corresponding connected Lie subgroup $G$ also acts transitively on $M$. (We depart from usual convention here and call a one-dimensional abelian ideal simple as well.) Furthermore, if $\mathfrak{h} = \mathfrak{h}' \cap \mathfrak{g}$, there is a subalgebra $\mathfrak{a} \subset \mathfrak{g}$, $\mathfrak{a} \cap \mathfrak{h} = 0$, $[\mathfrak{a}, \mathfrak{h}] = 0$ such that $\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{a}$, $\mathfrak{h}' = \mathfrak{h} \oplus \mathfrak{a}$, $\mathfrak{h} \subset \mathfrak{g}$ and $\mathfrak{a} \subset \mathfrak{h}'$ is embedded diagonally in $\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{a}$. Oniščik calls $(\mathfrak{g}', \mathfrak{h}')$ a simple extension of $(\mathfrak{g}, \mathfrak{h})$.

Since simple extensions are easy to enumerate, we assume now that $M = G/H$, where $\mathfrak{g}$ has at most two simple ideals. Oniščik in [23] has classified the rank 2 compact homogeneous spaces. We shall make use of these results.

We claim that no simple compact connected group can act transitively on $M$. To prove this, we may assume that $G$ is simply connected since $M$ is, and since $\pi_2 M \approx \mathbb{Z}$, it follows that $\pi_1 H = \mathbb{Z}$. But $\pi_3 G = \mathbb{Z}$ and $\pi_2 H = 0$, so $\pi_3(G/H)$ cannot be $\mathbb{Z} \oplus \mathbb{Z} = \pi_3 M$ if $q = 1$. If $q > 1$, $\pi_3 M = \mathbb{Z}$, and the homotopy exact sequence for $H \to G \to G/H$ implies that $\pi_3 H \to \pi_3 G$ is the zero map. This, together with $\pi_1 H = \mathbb{Z}$, shows that $H = S^1$. Now $\pi_k G \cong \pi_k(G/H)$ for $k \geq 3$, so $r(G) = 2$. But $r(G)$ is the rank of $G$ as a Lie group and so $G = \mathrm{SU}(3)$, $\mathrm{Sp}(2)$, or $G_2$. Since $\pi_k(M) = \pi_k(S^3) \oplus \pi_k(S^{2q+1}) = \pi_k(G)$ for $k \geq 3$, it follows easily that all three possibilities for $G$ can be ruled out.

It follows easily from $\pi_1 M = 0$ and the above that $\mathfrak{g}$ is semisimple. There are now two further cases.

(a) *$M$ is a product of two rank* 1 *homogeneous spaces.* From the rational cohomology of $M$, one sees that if $M = M_1 \times M_2$, then $H^*(M_1; Q) = H^*(S^2; Q)$ and $H^*(M_2; Q) = H^*(S^{2q+1}; Q)$. By Table 2 in [23] and the facts $\pi_2 M = \mathbb{Z}$ and $H^4(M; \mathbb{Z}) = \mathbb{Z}/|l|^2$, it follows that $M = S^2 \times S^{2q+1}$, where $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$ and $S^{2q+1}$ can be written as any homogeneous space of the simple groups which can act transitively on it.

(b) *The cases given by Theorem* 11 *in* [23]. By dimension reasons we need only consider I-IV. II to IV can be ruled out by comparing the homotopy groups of $M$ and $G/H$ as we did when $G$ was assumed to be simple. The remaining case is the following: $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{u}_{12} \oplus \mathfrak{h}_2$ with $\mathfrak{h}_i \subset \mathfrak{g}_i$, $\mathfrak{u}_{12}$ embedded diagonally in $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, and $r(G_i/H_i) = 1$. Again by comparing homotopy groups we see that the only possibilities are

$(\mathfrak{su}(2) \oplus \mathfrak{su}(q+1), \mathbb{R} \oplus \mathfrak{su}(q))$, $q \geq 1$, and $(\mathfrak{su}(2) \oplus \mathfrak{sp}(\frac{1}{2}(q+2)), \mathbb{R} \oplus \mathfrak{sp}(\frac{1}{2}q))$, $q$ even.

The proposition follows immediately from examining simple extensions of the possibilities in (a) and (b). q.e.d.

If we have a subfamily of $\{M_{k,l}^{1,q}\}$ which consists of diffeomorphic manifolds, one may consider $G'_{kl}$-homogeneous metrics on $M_{k,l}^{1,q}$. It is clear that these make $M_{k,l}^{1,q} \to P^1\mathbb{C} \times P^q\mathbb{C}$ into a Riemannian submersion with totally geodesic fibers, and are special cases of the metrics we constructed in §1. So the Einstein metrics in (1.4) for these special cases are $G'_{kl}$-homogeneous. (See (3.1) for a more general result.) By (2.5) the connected isometry group is $G'_{kl}$, hence by (2.4) we see that these Einstein metrics are isometric iff $|k| = |k'|$ and $|l| = |l'|$ for $q > 1$, and iff $(|k|, |l|)$ is a permutation of $(|k'|, |l'|)$ for $q = 1$. This already gives the existence of infinitely many nonisometric Einstein metrics with positive scalar curvature on the manifolds in (2.3).

**Remark.** While we are not studying homogeneous Einstein metrics in this paper, we note that the uniqueness statement in (1.4) and the fact that the Einstein metrics in (1.4) for the manifolds $M_{k,l}^{p,q}$ are $G'_{k,l}$-homogeneous yield examples of homogeneous spaces whose scalar curvature functional on the space of volume 1 *homogeneous* metrics have a unique critical point. On the other hand, the scalar curvature functional is neither bounded from above nor from below (see §2 of [32]).

## 3. The cohomogeneity of the Einstein metric

If $(M, g)$ is a Riemannian manifold, we define its cohomogeneity, denoted by $\mathrm{coh}(M, g)$, to be the codimension of a principal orbit of the action of the isometry group $I(M, g)$ on $M$. Then $(M, g)$ is Riemannian homogeneous iff $\mathrm{coh}(M, g) = 0$. In this section we study the cohomogeneity of the Einstein metrics we constructed in §1.

**(3.1) Proposition.** *In Theorems* (1.4) *and* (1.10), *if the base manifolds* $(M_j, g_j)$ *are homogeneous Kähler-Einstein manifolds, then the Einstein metrics* $(\ ,\ )_P$ *constructed on the total spaces are also homogeneous. More generally, the action of the connected isometry group* $G^*$ *of the metric* $\perp x_j g_j$ *on the base lifts to an effective action of a suitable finite covering group* $G$ *on* $P$ *which commutes with the* $T^r$ *action. Furthermore,* $G \times T^r$ *acts via isometries of* $(\ ,\ )_P$ *and hence* $\mathrm{coh}(P, (\ ,\ )_P) \leq \sum_{j=1}^m \mathrm{coh}(M_j, g_j)$.

*Proof.* Since $(M_j, x_j g_j)$ is Kähler the connected isometry group $G_j^* = I^0(M_j, g_j)$ is a subgroup of the identity component of the group of automorphisms of $(M_j, J_j)$ (see [2, 2.125]), and so $G_j^*$ acts in a natural way on $T^{0,1}(M_j)$ preserving the hermitian inner product induced by $g_j$ and $J_j$. This in turn induces an action of $G_j^*$ on the canonical line bundle $\Lambda^{(\frac{1}{2})n_j}(T^{0,1}(M_j))$ which again preserves the induced hermitian metric. Hence there is an effective action of $G_j^*$ on $P_j^*$, the associated principal $S^1$ bundle of the canonical line bundle, which commutes with the $S^1$ action. The principal $S^1$ bundle $P_j$ corresponding to $\alpha_j$ is the universal cover of $P_j^*$, and the covering transformations are just $\mathbb{Z}/q_j\mathbb{Z} \subset S^1$. The action of $G_j^*$ on $P_j^*$ is easily seen to lift to an effective action of a suitable finite cover $G_j$ of $G_j^*$ on $P_j$. Moreover, this action of $G_j$ commutes with the right $S^1$ action on $P_j$ since the $S^1$ action on $P_j$ covers the $S^1$ action on $P_j^*$.

Consider $\pi: P_1 \times \cdots \times P_m \to M_1 \times \cdots \times M_m$. Let $G = \prod_{j=1}^m G_j$. $G$ acts effectively on $P_1 \times \cdots \times P_m$, and commutes with the action of $T^m \approx (S^1)^m$ on $P_1 \times \cdots \times P_m$. We saw in §2 that the principal $T^r$ bundle $P$ over $M_1 \times \cdots \times M_m$ with characteristic classes $\beta_k = \sum_{j=1}^m b_{kj}\pi_j^*\alpha_j$, $1 \le k \le r$, can be described as the quotient $(P_1 \times \cdots \times P_m)/T^{m-r}$ for an appropriate $(m-r)$-dimensional subtorus $T^{m-r} \subset T^m$, with its $T^r$-action given by an isomorphism $T^r \approx T^m/T^{m-r}$. So there is an almost effective action of $G \times T^r$ on $P$. It follows that the codimension of a principal orbit of the $G \times T^r$ action on $P$ is less than or equal to the codimension of a principal orbit of the $G^* = \prod_{j=1}^m G_j^*$ action on $M_1 \times \cdots \times M_m$.

Finally, we consider Riemannian metrics on $P$. Recall that in §1 we constructed a principal connection $\theta$ on $P$ with curvature $d\theta = \pi^*\eta$, where $\eta = (\eta_1, \cdots, \eta_r)$ and $\eta_k = \sum_{j=1}^m b_{kj}\pi_j^*\omega_j$. Since $\omega_j$ is the Kähler form of $(M_j, g_j)$, $\eta$ is a $G^*$-invariant harmonic 2-form (with respect to any product metric $\perp (x_j g_j)$). Using the normalized Haar measure of $G$, we may average $\theta$: $\theta' = \int_G g^*\theta\, d\mu(g)$. Then $d\theta' = \pi^*\eta$, and so, as in §1, $\theta'$ is gauge-equivalent to $\theta$. Any Riemannian metric on $P$ formed by using $\perp x_j g_j$ on the base, a left-invariant metric on $T^r$, and $\theta'$ will be $G$-invariant. Hence a finite quotient of $G \times T^r$ is a subgroup of the connected isometry group of the Einstein metric $(\ ,\ )_P$ we constructed in §1. q.e.d.

Next we will examine when $\mathrm{coh}(P, (\ ,\ )_P) \ge \sum_{j=1}^m \mathrm{coh}(M_j, g_j)$, or, equivalently, when all isometries of $(P, (\ ,\ )_P)$ come from isometries of the base. This is of course not true for any Riemannian submersion, as can be seen in the case of the Hopf fibration $S^3(1) \to S^2(\frac{1}{2})$. Nevertheless, we suspect that this is the case for all of our principal torus bundles except the degenerate cases where $P$ is a product of $r$ $P_j$'s and $(m-r)$ $M_i$'s. However, we succeed in proving this only when the topology of the torus bundles is sufficiently complicated.

The idea of the proof is the following. Since isometries of $( \, , \, )_P$ must take geodesics to geodesics, we assume that $( \, , \, )_P$ has the property that the shortest nontrivial closed geodesics lie in the fibers and span the fibers in the sense that for every $p \in P$ the tangent vectors of all shortest closed geodesics passing through $p$ span the tangent space of the fiber. It follows that every isometry must map fiber to fiber, and hence descends to an isometry of the base. This implies that $\mathrm{coh}(P, ( \, , \, )_P) \geq \mathrm{coh}(M, \perp x_j g_j)$. If the topology of the bundle is complicated enough, then the Einstein condition implies that the diameter of the torus is small and hence the vertical tangent space must be spanned by short closed geodesics. The difficult part of the proof is showing that the closed geodesics of $( \, , \, )_P$ which do not lie in the fibers have to be longer.

We recall first some generalities about geodesics on the total space of a Riemannian submersion with totally geodesic fibers. (See, for example, [20] and [2].)

Let $(F, g_F) \to (M, g) \xrightarrow{\pi} (B, g_B)$ be such a submersion with O'Neill tensor $A$. (We assume of course that $g$ is complete.) For any curve $\sigma$ in $B$ and $x$ in $M$ above $\sigma(0)$, there is a unique horizontal lift $\tilde{\sigma}$ of $\sigma$ to $x$. The horizontal lifts of $\sigma$ induce an isometry of $\pi^{-1}(\sigma(0))$ onto $\pi^{-1}(\sigma(s))$ denoted by $\tau_\sigma|[0,s]$. Let $\gamma(s)$ be a unit speed geodesic in $P$. If $\dot{\gamma}(0)$ is vertical, then $\gamma$ is a geodesic in a fiber for the metric $g_F$. If $\dot{\gamma}(0)$ is horizontal, then $\dot{\gamma}(s)$ is horizontal for all the $s$ and so $\gamma$ projects to a geodesic of $(B, g_B)$. We are interested in the case where $\dot{\gamma}(0)$ is neither vertical nor horizontal. Let $c(s) = \pi \circ \gamma(s)$ and $\alpha(s) = \tau^{-1}_{c|[0,s]}(\gamma(s))$. If $\beta(t,s) = \tau_{c|[0,s]|}(\alpha(t))$, then $\gamma(s) = \beta(s,s)$ and $\dot{\gamma} = \beta_t + \beta_s$. Notice that for a fixed $t$, $\beta(t,s)$ is just the horizontal lift of $c$ to $\alpha(t)$, and for a fixed $s$, $\beta(t,s)$ is just the image of $\alpha$ in $\pi^{-1}(c(s))$ by the isometry $\tau$. Since $[\beta_t, \beta_s] = 0$, the geodesic equation for $\gamma$ gives $\nabla_{\beta_t}\beta_t + 2\nabla_{\beta_t}\beta_s + \nabla_{\beta_s}\beta_s = 0$. Note that $\nabla_{\beta_s}\beta_s$ is horizontal because $\mathscr{V}\nabla_{\beta_s}\beta_s = A_{\beta_s}\beta_s$ and for horizontal vectors $X$, $Y$, $A_X Y = -A_Y X$. Since the fibers are totally geodesic, $\nabla_{\beta_t}\beta_t$ is vertical and $\nabla_{\beta_t}\beta_s$ is horizontal. For a vertical vector $V$ we define the skew-symmetric endomorphism $L_V$ of the horizontal space by $(L_V(X), Y)_P = -(A_X Y, V)_P$. Hence $\nabla_{\beta_t}\beta_s = L_{\beta_t}(\beta_s)$ and the geodesic equation splits into

$$\nabla_{\beta_t}\beta_t = 0 \quad \text{and} \quad \nabla_{\beta_s}\beta_s + 2L_{\beta_t}(\beta_s) = 0. \tag{3.2}$$

The first equation says that $\alpha(s)$ is a geodesic in the fiber. It follows that $|\beta_t|$, $|\beta_s|$ and hence $|\dot{c}(s)|$ are constant. The second equation in general does not descend to an equation on $B$ for $c$.

Next, we specialize to a Riemannian submersion of the form $(P, (\ ,\ )_P) \xrightarrow{\pi} (B, (\ ,\ )_B)$ where $P$ is a principal $T^r$ bundle, and $(\ ,\ )_P$ is constructed using a principal connection $\theta$, $(\ ,\ )_B$, and a left-invariant metric $(h_{kl})$ on $T^r$. Let $d\theta = \pi^*\eta$ and $\eta = \sum_{k=1}^{t} \eta_k e_k$, where $\{e_1, \cdots, e_r\}$ is the standard basis for $\mathfrak{t} \approx \mathbb{R}^r$, as in §1. Let $W \in \mathfrak{t}$, $W = \sum w_k e_k$, and $\overline{W}$ be the vertical field on $P$ induced by the one-parameter group $\exp(tW)$. Then

$$(L_{\overline{W}}(X), Y)_P = \frac{1}{2}(\eta(\pi_* X, \pi_* Y), W)_{T^r} = \frac{1}{2}\sum_{k,l} h_{kl}\eta_k(\pi_* X, \pi_* Y) w_l.$$

Let $\eta_k'$ be the linear operator associated to $\eta_k$ via $(\ ,\ )_B$. Then the second equation in (3.2) becomes

$$\nabla_{\dot c}\dot c + \sum_{k,l} h_{kl} a_l \eta_k'(\dot c) = 0, \tag{3.3}$$

where $\theta(\beta_t) = \sum_{k=1}^{r} a_k e_k$. Note that the $a_k$ are constants because $\theta(\beta_t(s,s)) = \theta(d\tau_{c|[0,s]}(\dot\alpha(s))) = \theta(\dot\alpha(s)) = \theta(\dot\alpha(0))$, so that $\alpha$ enters into (3.3) only through the constants $a_k$ given by $\theta(\dot\alpha(0)) = \sum_{k=1}^{r} a_k e_k$. Furthermore, the least period of the closed geodesic $\gamma$ is, in general, a multiple of the least period of $c$. We will now bound the minimal period of periodic solutions of (3.3) from below under suitable hypotheses for the principal $S^1$ and torus bundles considered in (1.4) and (1.10). We assume that $(\ ,\ )_P$ is the Einstein metric with $E = 1$ we constructed. We will treat the case of $S^1$ bundles in detail first, and then indicate the changes for the case of $T^r$ bundles, $r \geq 2$.

**(3.4) Theorem.** *Let* $\pi\colon (P, (\ ,\ )_P) \to (M_1 \times \cdots \times M_m, \perp x_j g_j)$ *be the principal* $S^1$ *bundle with Euler class* $\sum_{j=1}^{m} b_j \pi_j^* \alpha_j \neq 0$ *in* (1.4), *where* $(\ ,\ )_P$ *is the Einstein metric with* $E = 1$. *If* $\sum b_j^2$ *is sufficiently large, then*

$$\operatorname{coh}(P, (\ ,\ )_P) = \sum_{j=1}^{m} \operatorname{coh}(M_j, g_j).$$

*Proof.* In view of (3.1), it is only necessary to prove that $\operatorname{coh}(P, (\ ,\ )_P) \geq \sum_{j=1}^{m} \operatorname{coh}(M_j, g_j)$, which would follow if the shortest nontrivial closed geodesics of $(\ ,\ )_P$ are the fiber circles. Let $\gamma$ be a unit speed closed geodesic of $(\ ,\ )_P$. Its projection $c$ onto the base consists of closed curves $c_i$ in $M_i$. Its projection $\alpha$ to the fiber through $\gamma(0)$ has $\theta(\dot\alpha(0)) = a \in \mathbb{R}$, where $a\rho = |\dot\alpha(0)|_F$ and the fibers have length $2\pi\rho$. Since

$$\eta(X, Y) = (\eta'(X), Y)_B = \sum_j b_j \pi_j^* \omega_j(X, Y) = \sum_j b_j g_j(J_j(\pi_j^* X), \pi_j^* Y),$$

it follows that $\eta'(X) = \sum_j (b_j/x_j) J_j(\pi_j^* X)$. Hence (3.3) becomes

$$\nabla_{\dot c_j}\dot c_j + \rho|\dot\alpha(0)|_F (b_j/x_j) J_j(\dot c_j) = 0, \qquad 1 \leq j \leq m. \tag{3.5}$$

It is possible that some of the $c_j$ are point curves, but since $\gamma$ is not vertical, the least period of $\gamma$ is divisible by the least periods of all those $c_j$ which do not reduce to point curves. Observe that since $|\dot{\gamma}| = 1 = |\dot{\alpha}|^2 + \sum_{j=1}^{m} |\dot{c}_j|^2$, (1.8) implies that there exists a constant $A > 0$, depending only on the $n_j$ and $q_j$, such that $|\dot{c}_j|_{g_j} \leq A$, $1 \leq j \leq m$, and $2|\dot{\alpha}|_F \leq A$ for all nonvertical unit speed closed geodesics $\gamma$ of $(\ ,\ )_P$. Therefore, if $\sum b_j^2$ is large enough, $\rho$ will be arbitrarily small by (1.5), and our proof will be complete if we can show that if $2|\dot{\alpha}|_F \leq A$ then the least period of all nontrivial periodic solutions of (3.5) with $|\dot{c}_j(0)|_{g_j} \leq A$ is bounded from below by a positive constant. In fact, (1.5) and (1.8) imply that

$$\rho\left(\frac{b_j}{x_j}\right) = \frac{b_j}{x_j}\left(\sum_j n_j \left(\frac{b_j}{x_j}\right)^2\right)^{1/2} (4E)^{1/2} \leq 2.$$

So it remains to prove

**(3.6) Lemma.** *Consider the equation $\nabla_{\dot{c}}\dot{c} + \sigma J(\dot{c}) = 0$ on a compact Kähler manifold $(N, g)$, where $\sigma$ is a positive constant and $J$ is the complex structure. Given $A > 0$, there is an $L_0 > 0$ such that for all $\sigma \leq A$ and all nontrivial periodic solutions of the equation with $|\dot{c}(0)| \leq A$, the least period of $c$ is greater than or equal to $L_0$.*

*Proof.* The equation is not invariant under reparametrization. Indeed, if $c_\lambda(s) = c(\lambda s)$, then $c_\lambda$ satisfies $\nabla_{\dot{c}_\lambda}\dot{c}_\lambda + \lambda\sigma J(\dot{c}_\lambda) = 0$. So we consider the equation

$$\nabla_{\dot{y}}\dot{y} + J(\dot{y}) = 0, \tag{3.7}$$

and then translate the results back using $c(s) = y(\sigma s)$ and $\dot{c}(0) = \sigma\dot{y}(0)$.

For $v \in T_pN$, we let $y_v(t)$ be the unique solution of (3.7) with initial conditions $y_v(0) = p$ and $\dot{y}_v(0) = v$. We will prove that

(*) there is a neighborhood $U$ of the zero section of $TN$ such that the least period of all nontrivial periodic solutions $y_v$ of (3.7) with $v \in U$ are bounded from below by a positive number.

This fact implies that for any compact set $K \subset TN$, there exists a positive lower bound for the least period of all nontrivial periodic solutions $y_v$ of (3.7) with $v \in K$.

On the other hand, if $v_i \in TN$, $|v_i| \to \infty$, and $y_{v_i}$ are periodic solutions of (3.7), then the unit speed reparametrization $y_{v_i}^*(t) = y_{v_i}(t/|v_i|)$ satisfies

$$\nabla_{\dot{y}_{v_i}^*}\dot{y}_{v_i}^* + \frac{1}{|v_i|}J(\dot{y}_{v_i}^*) = 0.$$

As $i \to \infty$, $y^*_{v_i}$ converges to a nontrivial closed geodesic of $(N, g)$. Since the lengths of the nontrivial closed geodesics of $(N, g)$ are bounded from below by a positive constant, there are positive constants $K$ and $L_1$ such that a periodic solution $y_v$ of (3.7) with $|v| \geq K$ has least period $\geq L_1/|v|$. By the previous paragraph, there is a positive constant $L_2$ such that a nontrivial periodic solution $y_v$ of (3.7) with $|v| \leq K$ has least period $\geq L_2$.

Set $L_0 = \min((L_1/A), (L_2/A))$. If $\sigma \leq A$ and $c$ is a nontrivial periodic solution of (3.6) with $|\dot{c}(0)| \leq A$, let $y(t) = c(t/\sigma)$. Then $|\dot{y}(0)| = |\frac{1}{\sigma}\dot{c}(0)|$ and $y$ is a nontrivial periodic solution of (3.7). The least period of $c = \frac{1}{\sigma}$ (least period of $y$) $\geq \frac{1}{\sigma}\min(L_2, L_1\sigma/|\dot{c}(0)|) \geq \min((L_1/A), (L_2/A)) = L_0 > 0$. Thus the proof of (3.6) is complete once we prove (*).

For this, choose $p \in N$ and for any time $t$ consider the map $f_{t,p}(v) = y_v(t)\colon T_pN \to N$. Exactly as in the analysis of the differential of the exponential map, we consider the variation field $Z$ along a solution $y$ of (3.7) for a one-parameter variation of $y$ through solutions. The analog of the Jacobi equation is $\nabla_{\dot{y}}\nabla_{\dot{y}}Z + R(Z, \dot{y})\dot{y} + (\nabla_Z J)\dot{y} + J(\nabla_{\dot{y}}Z) = 0$. Now

$$(df_{t,p})_0(v) = \tfrac{d}{ds}|_{s=0} f_{t,p}(sv) = \tfrac{d}{ds}|_{s=0} y_{sv}(t) = Z(t),$$

where we regard $y_{sv}(t)$ as a variation of the constant solution $y(t) = p$ with variational field $Z$. So the equation $Z$ satisfies is obtained from the modified Jacobi equation by passing to the limit as $s \to 0$. Clearly, the tensorial terms in $\dot{y}$ drop out, and $\nabla_{\dot{y}}Z$ (resp. $\nabla_{\dot{y}}\nabla_{\dot{y}}Z$) tends to $Z'$ (resp. $Z''$) with $Z(0) = 0$ and $Z'(0) = v$. So $Z'' + J(Z') = 0$. We can compute $\det((df_{t,p})_0)$ by using an orthonormal basis $\{v_i\}$ of eigenvectors of $J \otimes \mathbb{C}$ and explicitly solving $Z'' + J(Z') = 0$ with $Z(0) = 0$ and $Z'(0) = v_i$. Then

$$\det((df_{t,p})_0) = \prod_i \langle (df_{t,p})_0(v_i), v_i \rangle = \prod_i \frac{1}{\lambda_i}(1 - e^{-\lambda_i t}).$$

Since the eigenvalues of $J$ are $\pm\sqrt{-1}$ with multiplicity equal to the complex dimension of $N$, there is an interval $[t_1, t_2]$ with $t_2 \geq 2t_1 > 0$ such that if $t \in [t_1, t_2]$, $\det((df_{t,p})_0) \neq 0$. Using first the inverse function theorem, and then the compactness of $N$ and $[t_1, t_2]$, we can find a neighborhood $U$ of the zero section of $TN$ such that $f_{t,p}$ is injective on $T_PN \cap U$ for all $p \in N$ and all $t \in [t_1, t_2]$. Now suppose that $v \in U$. If $y_v$ is a nontrivial periodic solution of (3.7) with least period $L < t_1$, then there is an integer $n$ so that $nL$ is a period of $y_v$ lying in $[t_1, t_2]$ because $(t_1 - t_2)/L > 1$, so the interval $[t_1/L, t_2/L]$ must contain an integer. Hence $t_2 > 0$ is a lower bound for $L$, proving (*). q.e.d.

Next, we turn to principal torus bundles $P$ over $M_1 \times \cdots \times M_m$. We assume that the characteristic classes of $P$ are $\beta_k = \sum b_{kj}\pi_j^*\alpha_j$, $1 \leq k \leq r$,

$r \geq 2$, and that $(\ ,\ )_P$ is the Einstein metric with $E = 1$ in (1.10). For a unit speed closed geodesic of $(\ ,\ )_P$ we define $c$ and $\alpha$ as before. Aside from finding a positive lower bound for the least period of nonvertical closed geodesics $\gamma$, we must also ensure that all the eigenvalues of the left-invariant metric $(h_{kl})$ determined by the Einstein condition could be made sufficiently small by making $\beta_i$ "complicated" in a suitable sense.

Equation (3.3) becomes

$$\nabla_{\dot{c}_j}\dot{c}_j + \sum_{k,l} h_{kl}a_l(b_{kj}/x_j)J_j(\dot{c}_j) = 0, \qquad 1 \leq j \leq m.$$

We analyse first the constant $\sum_{k,l} a_l h_{lk}(b_{kj}/x_j) = \sigma_j$. Let $u_j$ be the vector $\sum_{k=1}^r (b_{kj}/x_j)e_k$ and $a$ be the vector $\sum_{k=1}^r a_k e_k$. Then $|\sigma_j| = |(a, u_j)_{T^r}| \leq |a|_{T^r}|u_j|_{T^r}$. But (1.12) can be rewritten as $\frac{1}{2}|u_j|^2_{T^r} = (q_j/x_j) - 1 \leq 1$ by (1.16). So $|\sigma_j| \leq 2|a|_{T^r}$. Because

$$|\dot{\gamma}| = 1 = |\dot{\alpha}(0)|^2 + \sum_j |\dot{c}_j(0)|^2 = |a|^2_{T^r} + \sum_j x_j|\dot{c}_j(0)|^2_{g_j},$$

there is again a constant $A > 0$ depending only on the $M_j$ and not on $P$ such that $2|a|_{T^r} \leq A$ and $|\dot{c}_j(0)|_{g_j} \leq A$, $1 \leq j \leq m$. We may therefore appeal to Lemma (3.6) again to get the lower bound for the least period.

It remains to examine when the eigenvalues of $(h_{kl})$ can be made small. Since $(h_{kl})$ is positive definite, this is equivalent to making $\mathrm{tr}(H)$ small. By (1.14),

$$\begin{aligned}\mathrm{tr}(H) &= \frac{4\sum_k |v_1 \wedge \cdots \wedge \check{v}_k \wedge \cdots \wedge v_r|^2}{|v_1 \wedge \cdots \wedge v_r|^2} \\ &= \frac{4C\sum_k |w_1 \wedge \cdots \wedge \check{w}_k \wedge \cdots \wedge w_r|^2}{|w_1 \wedge \cdots \wedge w_r|^2}\end{aligned}$$

for some constant $C$ depending only on the $M_j$, and $w_k$ are the rows of $(b_{ij})$ viewed as vectors in $\mathbb{R}^m$. Dividing by the lengths, we get

$$\mathrm{tr}(H) = 4C\sum_k \frac{1}{|w_k|^2}\frac{|\varepsilon(w_1) \wedge \cdots \wedge \check{\varepsilon}(w_k) \wedge \cdots \wedge \varepsilon(w_r)|^2}{|\varepsilon(w_1) \wedge \cdots \wedge \varepsilon(w_r)|^2},$$

where $\varepsilon(w_k)$ is the unit vector $w_k/|w_k|$. Since we assume that $(b_{ij})$ has maximal rank, the vectors $\varepsilon(w_1), \cdots, \varepsilon(w_r)$ are linearly independent, and so $|\varepsilon(w_1) \wedge \cdots \wedge \check{\varepsilon}(w_k) \wedge \cdots \wedge \varepsilon(w_r)|$ and $|\varepsilon(w_1) \wedge \cdots \wedge \varepsilon(w_r)|$ are never 0. Consider the manifold of (ordered) $r$-frames of unit vectors in $\mathbb{R}^m$, $U_r(\mathbb{R}^m)$. The $r$ functions

$$f_k(\varepsilon_1, \cdots, \varepsilon_r) = \frac{|\varepsilon_1 \wedge \cdots \wedge \check{\varepsilon}_k \wedge \cdots \wedge \varepsilon_r|^2}{|\varepsilon_1 \wedge \cdots \wedge \varepsilon_r|^2}$$

are continuous, so they are bounded from above and below on any compact subset $K$ of $U_r(\mathbb{R}^m)$. If $K$ contains rational $r$-frames (i.e., frames whose vectors have rational coordinates) with arbitrarily large denominators after being put in lowest terms, then the corresponding $\operatorname{tr}(H)$ can be made as small as we please. Hence, we have

**(3.8) Theorem.** *Let $\pi\colon (P, (\ ,\ )_P) \to (M_1 \times \cdots \times M_m, \perp x_j g_j)$ be the principal $T^r$ bundle with characteristic classes $\beta_k = \sum_j b_{kj}\pi_j^*\alpha_j$ as in* (1.10), *where $(\ ,\ )_P$ is the Einstein metric with $E = 1$, $r \geq 2$. For every sequence $P_n$ of these principal $T^r$ bundles with $|w_k^{(n)}| \to \infty$, $1 \leq k \leq r$, and $(\varepsilon(w_1^{(n)}), \cdots, \varepsilon(w_r^{(n)}))$ lying in a compact subset of $U_r(\mathbb{R}^m)$, there is an $N$ such that for all $n \geq N$,*

$$\operatorname{coh}(P_n, (\ ,\ )_{P_n}) = \sum_{j=1}^{m} \operatorname{coh}(M_j, g_j).$$

**Remark.** As an example to show that some condition is necessary in (3.8), consider

$$B = (b_{kj}) = \begin{pmatrix} 1 & 0 & b_3 & b_4 & \cdots & b_m \\ 0 & 1 & b_3 & b_4 & \cdots & b_m \end{pmatrix}.$$

The corresponding $T^2$ bundles are inequivalent under the action of $\mathrm{SL}(2, \mathbb{Z})$. However,

$$\frac{\sum_k |w_1 \wedge \cdots \wedge \check{w}_k \wedge \cdots \wedge w_r|^2}{|w_1 \wedge \cdots \wedge w_r|^2} = 2\frac{1 + \sum b_j^2}{1 + 2\sum b_j^2},$$

which tends to 1 as $\sum b_j^2 \to \infty$.

Finally, we indicate how one proves Corollary 6 in the introduction. First, we observe that if we have a deformation of the Kähler-Einstein metric $g_i$ on $M_i$, then the solution $\rho, x_i$ of the Einstein condition (1.9) does not depend on $g_i$ as long as the corresponding Kähler class $\omega_i$ satisfies $[\omega_i] = 2\pi\alpha_i$. Hence the length of the fiber is constant and can be chosen arbitrarily small by choosing $\sum_i b_i^2$ arbitrarily large. Second, we observe that the uniform lower bound in Lemma (3.6) still exists for a smooth compact family of complex structures on $N$ (with corresponding Kähler metrics). Hence, the fibers will be the shortest nontrivial closed geodesics for the Einstein metrics on $P$. This implies that two metrics on $P$ are isometric iff the corresponding metrics on the base are isometric, thus proving Corollary 6.

## 4. Geometric applications

**I. Einstein constants.** We first examine the Einstein constant in our examples.

**(4.1) Theorem.** *Consider a sequence of principal $T^r$ bundles as in* (1.10) *and normalize the Einstein metrics to have volume* 1. *If the bundles are inequivalent under the action of* $\mathrm{SL}(r, \mathbb{Z})$ *on principal $T^r$ bundles, then the Einstein constants converge to* 0.

*Proof.* The volume of $P$ is the product of the volume of the base and the volume of the fiber. If we normalize the metrics so that $E = 1$, then (1.16) implies that the volume of the base is universally bounded from above and away from 0. So $\mathrm{vol}(P)$ behaves like the volume of the fiber, i.e., $\det(h_{ij})$. But (1.13) implies that $\det(h_{ij})$ behaves like $1/|w_1 \wedge \cdots \wedge w_r|^2$, where $w_i$ is the $i$th row in the matrix $B = (b_{ij})$, viewed as a vector in Euclidean $m$-space. Hence $\mathrm{vol}(P) \to 0$ iff $|w_1 \wedge \cdots \wedge w_r| \to \infty$. Or, equivalently $E \to 0$ iff $|w_1 \wedge \cdots \wedge w_r| \to \infty$.

We claim that $|w_1 \wedge \cdots \wedge w_r|$ has only $\infty$ as an accumulation point for a sequence of principal $T^r$ bundles as in (1.10) which are inequivalent under the action of $\mathrm{SL}(r, \mathbb{Z})$. Recall from §1 that $A \in \mathrm{SL}(r, \mathbb{Z})$ acts on $B$ by multiplying $B$ on the left by $(A^t)^{-1}$. Assume that $|w_1 \wedge \cdots \wedge w_r|$ has a positive finite accumulation point, so that for some subsequence $|w_1 \wedge \cdots \wedge w_r|$ is bounded. Notice that $|w_1 \wedge \cdots \wedge w_r|$ is unaffected if we multiply $B$ by $(A^t)^{-1}$. By interchanging the $M_i$ if necessary, we can assume that the first $r$ columns of $B$ form a nonsingular matrix $C$. The determinant of $C$ must remain bounded, for otherwise it would contradict the boundedness of $|w_1 \wedge \cdots \wedge w_r|$. So we may assume, after passing to a subsequence, that $\det C$ is constant. But integer matrices with a fixed determinant fall into only finitely many orbits of $\mathrm{SL}(r, \mathbb{Z})$. Hence we can further assume that $C$ is a constant matrix. The boundedness of $|w_1 \wedge \cdots \wedge w_r|$ implies that the coefficients of the terms $e_1 \wedge \cdots \wedge \check{e}_i \wedge \cdots \wedge e_r \wedge e_j$ in $w_1 \wedge \cdots \wedge w_r$ are bounded for all $1 \leq i \leq r$, $r+1 \leq j \leq m$. But then the corresponding entries $B_{ij}$ of $B$ are all bounded, a contradiction.

**II. Collapsing.** In [10, Chapter 8], Gromov defined the following concept of collapsing. If $(X_i, g_i)$ is a sequence of Riemannian manifolds, they are said to collapse if there exists a constant $\Lambda$ such that the sectional curvatures satisfy $|K(g_i)| \leq \Lambda$ and the injectivity radius tends to 0. Consider a sequence of principal $T^r$ bundles as in (1.10) with Einstein metrics normalized so that $E = 1$ and such that the bundles are inequivalent under the action of $\mathrm{SL}(r, \mathbb{Z})$. By (4.1) the volumes of the metrics tend to 0, and so the injectivity radii go to 0. To show that these Einstein manifolds collapse, we need to derive a universal curvature bound.

**(4.2) Proposition.** *Given the base $B = M_1 \times \cdots \times M_m$ and any principal torus bundles $P$ over $B$ as in* (1.10), *let us normalize the Einstein metrics on $P$ so that $E = 1$. Then there exists a constant $\Lambda$ depending only on $B$ such that the sectional curvatures of these Einstein metrics satisfy $|K| \leq \Lambda$.*

*Proof.* For a Riemannian submersion, one can express $\langle R(A,B)C,D\rangle$, where $A$, $B$, $C$, $D$ are either horizontal or vertical, in terms of the curvature of the base and fiber and in terms of the O'Neill tensors $A$ and $T$ and their covariant derivatives (see [2, 9.28]). It suffices to show that $\langle R(A,B)C,D\rangle$ is universally bounded, where $A$, $B$, $C$, $D$ are either horizontal or vertical unit vectors, because $\langle R(e_i,e_j)e_k,e_l\rangle$ is then bounded for an orthonormal basis $\{e_i\}$, which implies that the matrix of the curvature operator $\hat{R}$ with respect to the basis $\{e_i \wedge e_j\}$ has universally bounded coefficients, so that the sectional curvatures $K(A\wedge B) = \langle \widehat{R}(A\wedge B), A\wedge B\rangle$ for $|A\wedge B| = 1$ are universally bounded.

If we assume that the fibers are totally geodesic, then we need to bound, besides the sectional curvatures of the base and the fiber, the expressions $\langle A_X Y, A_Z Z'\rangle$, $\langle A_X U, A_Y V\rangle$, $\langle (\nabla_X A)_Y Z, V\rangle$ and $\langle (\nabla_U A)_X Y, V\rangle$, where $X, Y, Z, Z'$ are horizontal unit vectors and $U$, $V$ are vertical unit vectors. For the first two expressions, it is sufficient to bound $|A_X Y|^2$ since $\langle A_X U, Y\rangle = -\langle U, A_X Y\rangle$.

For metrics on the total space of a principal $G$-bundle as described at the beginning of §1, one easily shows that

$$((\nabla_X A)_Y Z, V)_P = -\tfrac{1}{2}((\nabla_X \Omega)(Y,Z), \theta(V))_G,$$

$$\begin{aligned}((\nabla_U A)_X Y, V)_P &= \tfrac{1}{2}([\theta(U), \Omega(X,Y)], \theta(V))_G + \tfrac{1}{2}(\Omega(X,Y), \theta(\nabla_U V))_G \\ &\quad - (L_U(X), L_V(Y))_P + (L_U(Y), L_V(X))_P \\ &= -\tfrac{3}{4}(\Omega(X,Y), \theta([U,V]))_G - (L_U(X), L_V(Y))_P \\ &\quad + (L_U(Y), L_V(X))_P,\end{aligned}$$

where $L_U(X)$ is the skew-symmetric operator defined by $(L_U(X), Y)_P = \frac{1}{2}(\theta(U), \Omega(X,Y))_G$, and the last equality holds iff $(\ ,\ )_G$ is bi-invariant.

For our torus bundles, the fibers are flat and by (1.16) the curvatures of the base metrics are universally bounded. Moreover, $\Omega = \pi^* \eta$, so $\nabla_X \Omega = \pi^*(\nabla_X \eta) = 0$, since $\eta$ is parallel with respect to any metric of the form $\perp x_i g_i$ on the base. Thus it is sufficient to bound $(\eta(X,Y), V)_G$ for unit vectors $X, Y \in TB$ and $V \in \mathfrak{g}$. But $E = 1$ and (1.2) implies that $\sum_{i,j,k}(\eta(X_i,X_j), v_k)_G^2 = 2\|\eta\|^2$ is universally bounded, and hence so is $(\eta(X,Y), V)_G$. q.e.d.

We return to the examination of the Einstein metrics on the total spaces of our sequence of torus bundles. Notice that by Myers' theorem, since $E = 1$, the diameters of these Einstein manifolds are bounded. So by [10, 5.3, the remark at the bottom of p. 66, and 3.8], we conclude that a subsequence of these manifolds converge in the Hausdorff topology to a limit metric space. Using (4.2) above and 8.30 in [10] we see that this limit metric space must have strictly smaller dimension. For circle bundles, since the fiber shrinks to a point, 3.5b in [10] implies that the limit metric space is $M_1 \times \cdots \times M_m$ with distance induced by some product metric $\perp x_i g_i$, $x_i > 0$, which, in general, need not be Einstein. In the case of torus bundles, the same is true if the diameter of the fiber goes to 0. For a condition which guarantees this, see (3.8). In general, a sequence of torus bundles converges to a torus bundle of lower dimension.

These examples of collapsing are also interesting in connection with a theorem of Fukaya [9] stating that if $(X_i, g_i)$ converge in the Hausdorff topology to a smooth manifold $(B, g)$, then there exist fibrations $\pi_i \colon X_i \to B$ whose deviation from being a Riemannian submersion goes to zero as $i \to \infty$. In our examples we can choose the total spaces of the circle bundles to be all diffeomorphic. Hence we obtain a sequence of metrics on a fixed manifold $M$ converging to $(B, g)$ such that the fibrations $\pi_i \colon M \to B$ are all topologically distinct.

**III. Pinching theorems.** In [4], [5] Cheeger defined a distance between Riemannian manifolds and proved some pinching theorems using this distance.

Let $(M, g)$ and $(M, g')$ be two Riemannian manifolds. Given an isometry $I \colon T_pM \to T_{p'}M'$ one defines a natural correspondence $\gamma \to \gamma'$ between broken geodesics on $M$ starting at $p$ and broken geodesics on $M'$ starting at $p'$ using parallel translation along the geodesics as in the Cartan-Ambrose-Hicks theorem. This defines an isometry $I_{\gamma|[0,t]} \colon T_{\gamma(t)}M \to T_{\gamma'(t)}M'$ using parallel translation along $\gamma$ and $\gamma'$. Let

$$\rho(M, M') = \inf_{p,p',I} \sup_{\gamma, L(\gamma)\le A} \|R - I_\gamma^{-1}(R')\|,$$

$$\rho^*(M, M') = \inf_{p,p',I} \sup_{\gamma, L(\gamma)\le A} \{\|R - I_\gamma^{-1}(R')\|, \|\nabla R - I_\gamma^{-1}(\nabla R')\|\},$$

where $\frac{1}{2}A$ is the supremum of the first conjugate distance over all geodesics in $M$. The Cartan-Ambrose-Hicks theorem states that for two compact simply connected manifolds $M$ and $M'$, $\rho(M, M') = 0$ iff $M$ is isometric to $M'$.

Cheeger then showed that if $M$ is a compact simply connected symmetric space whose cut-locus has codimension $\geq 3$, then there exists a $\delta > 0$ such that a simply connected compact manifold $M'$ with $\rho(M, M') < \delta$ is piecewise-linearly homeomorphic to $M$, and if $\rho^*(M, M') < \delta$, then $M'$ is diffeomorphic to $M$. Our examples will show that the condition that the cut-locus of $M$ has codimension $\geq 3$ is necessary in this and other theorems in [4], [5].

We illustrate this using $M_{k,l}^{p,q}$ equipped with the submersion metric induced by the product of the round sphere metrics on $S^{2p+1} \times S^{2q+1}$. To study $\rho$ and $\rho^*$, we need the following.

**(4.3) Proposition.** *Let $(M, g)$ be a compact Riemannian manifold and $\pi_i \colon (M, g) \to (B_i, g_i)$ be a sequence of Riemannian submersions with vertical distributions $\mathscr{V}_i$. If $\mathscr{V}_i \to \mathscr{V}_0$ as vector spaces, as $i \to \infty$, then $(B_i, g_i)$ converges to $(B_0, g_0)$ in the Cheeger distance $\rho$ and $\rho^*$.*

*Proof.* Fix a point $p \in M$ and let $p_i = \pi_i(p)$, $\mathscr{H}_i = \mathscr{V}_i^{\perp}$. The orthogonal projections $\mathscr{H}_i \to \mathscr{H}_0$ at $p \in M$ induce isometries $\tau_i \colon T_{p_i}B_i \to T_{p_0}B_0$. For any broken geodesics $\gamma_0$ at $p_0$, the corresponding broken geodesics $\gamma_i$ in $B_i$ starting at $p_i$ lift to broken $\mathscr{H}_i$ horizontal geodesics $\tilde{\gamma}_i$ in $M$ starting at $p$. These $\tilde{\gamma}_i$ converge, as $i \to \infty$, to a broken $\mathscr{H}_0$ horizontal geodesic in $M$. Furthermore, in $M$ as well as in $B_i$, the parallel translations along these geodesics also converge. By the O'Neill formulas, the curvature tensor on $B_i$ is described in terms of the curvature tensor on $M$ and the O'Neill tensor $A_i$ for $\pi_i$. Since $A_i \to A_0$ smoothly as $i \to \infty$, it follows that $\rho(B_i, B_0) \to 0$. One easily derives a similar formula for $\nabla R$, which then implies that $\rho^*(B_i, B_0) \to 0$. In fact the same is true for $\nabla^m R$ for all $m \geq 1$. q.e.d.

We now apply this to the manifolds $M_{k,l}^{p,q}$ with metric induced from the product metric on $S^{2p+1} \times S^{2q+1}$. If we fix $l$ and let $k \to \infty$, the vertical distributions for the circle actions on $S^{2p+1} \times S^{2q+1}$ converges to the one for $k = 1$, $l = 0$. Hence, $N_1^k = M_{k,l}^{p,q}$ converges to $N_1 = S^{2p+1} \times P^q\mathbb{C}$ in the metric $\rho^*$. Actually, since $N_1^k$ and $N_1$ are both homogeneous, there exists for every $x \in N_1^k$ and $y \in N_1$ an isometry $I \colon T_x N_1^k \to T_y N_1$ such that $\|R - I_\gamma^{-1}(R')\|$ is small. Similarly, if we fix $k$, then $N_2^l = M_{k,l}^{p,q}$ converges to $N_2 = P^p\mathbb{C} \times S^{2q+1}$. $N_1$ and $N_2$ are symmetric spaces whose cut-loci have portions of codimension 2. By (2.1) the cohomology ring of $N_2^l$ depends on $l$ and these examples therefore show that the condition on the cut-locus in Cheeger's theorem is necessary. For the $N_1^k$, the cohomology ring is the same for all $k$, but they all have distinct homeomorphism types if $p \geq 2$. If $p = 1$, we can choose a subsequence of $N_1^k$ which consists of diffeomorphic manifolds. For example, if $l = 1$ and $k$ is even then all $N_1^k$ are diffeomorphic to $S^2 \times S^{2q+1}$. Hence we obtain a sequence of metrics on $S^2 \times S^{2q+1}$ converging in the Cheeger distance $\rho^*$ to a product metric on $S^3 \times P^q\mathbb{C}$. By contrast, in the Hausdorff topology they converge towards a product metric on $S^2 \times P^q\mathbb{C}$. When $q = 1$, we obtain a sequence of irreducible metrics on $S^3 \times S^2$ converging in $\rho^*$ to a product metric on $S^3 \times S^2$.

In [5] Cheeger also proves that if $M$ is either a symmetric space whose cut-locus has codimension $\geq 3$ or a symmetric space such that some characteric number is nonzero, then there exist constants $i_0$, $\delta$ such that $\rho(M', M) < \delta$ implies that the injectivity radius of $M'$ is $\geq i_0$. In each case our examples show that the extra condition is necessary. He also proves that if $M$ is a symmetric space with some nonzero characteristic number, then there exists a $\delta > 0$ that $\rho(M, M') < \delta$ implies that $H^*(M'; k)$ is a subring of $H^*(M; k)$ for every field $k$. The examples $N_2^l$ show that this is false in general.

In [14], Katsuda showed that there is a positive constant $\delta(n, \Lambda, v, D)$ such that if $(M^n, g)$ is a complete Riemannian manifold of dimension $n$ with $|K(M)| \leq \Lambda^2$, $\mathrm{vol}(M) \geq v$, $\mathrm{diam}(M) \leq D$ and $|\nabla R| \leq \delta$, then $M$ is diffeomorphic to a locally symmetric space. Again, our examples show that the positive lower bound $v$ is a crucial hypothesis.

**Remark.** In [8] Eschenburg constructed the first nonhomogeneous examples of Riemannian manifolds with positive sectional curvature by exhibiting a sequence of free biquotient actions of $S^1$ on SU(3), isometric in some left-invariant metric, whose vertical spaces converge to the vertical spaces of the $S^1$ action given by some closed subgroup $S^1 \subset \mathrm{SU}(3)$. By Aloff-Wallach, the homogeneous spaces $\mathrm{SU}(3)/S^1$ admit metrics of positive sectional curvature, so the quotients of SU(3) by the biquotient actions eventually have positive sectional curvature also. Eschenburg also proved that his examples are not homotopy equivalent to any homogeneous space. (4.3) implies that the Eschenburg examples converge in the $\rho^*$ topology to the Aloff-Wallach examples, and hence a manifold $\rho^*$-close to a homogeneous space need not be homogeneous.

Finally, we consider the Einstein metrics with $E = 1$ on a sequence of spaces $M_{k_i,l_i}^{p,q}$. If $\{(l_i, -k_i)\} \subset \mathbb{R}^2$ converges to a ray through the origin with rational slope $-k_0/l_0 = \lim_{i\to\infty} -k_i/l_i$, where $k_0$, $l_0$ are in lowest terms, then by (1.9) it is clear that the solutions for $(b_1, b_2) = (k_i, l_i)$ also converge smoothly to a solution for $(b_1, b_2) = (k_0, l_0)$. Since the $M_{k_i,l_i}^{p,q}$ are homogeneous, it follows easily that they converge in the distance $\rho^*$ to

$M^{p,q}_{k_0,l_0}$ equipped with the Einstein metric. Again, one can compare the situation with that of Hausdorff convergence. On the other hand, if $\{(l_i, -k_i)\}$ converges to a ray through the origin with irrational slope, then something interesting also occurs. In this case, the closed circle subgroups of $T^2$ by which we divide $S^{2p+1} \times S^{2q+1}$ converge to a dense one-parameter subgroup, and so we can no longer form the quotient. However, locally the Einstein condition still makes sense, and the solutions depend smoothly on the slope $-k/l = -b_1/b_2$ by (1.9). Thus we obtain an example of an Einstein metric on an open set such that no covering of it admits an extension to a complete Einstein metric.

## References


[1] L. Bérard Bergery, *Sur de nouvelles variétés riemanniennes d'Einstein*, Publ. Inst. E. Cartan, No. 4, Nancy, 1982, 1–60.

[2] A. Besse, *Einstein manifolds*, Ergeb. Math. Grenzgeb, Band 10, Springer, Berlin, 1987.

[3] L. Castellani, R. D'Auria, & P. Fré, SU(3) × SU(2) × U(1) *from* $D = 11$ *supergravity*, Nuclear Phys. B **239** (1984) 610–652.

[4] J. Cheeger, *Comparison and finiteness theorems for Riemannian manifolds*, Thesis, Princeton University, 1967.

[5] ——, *Pinching theorems for a certain class of Riemannian manifolds*, Amer. J. Math. **91** (1969) 807–834.

[6] J. D'Atri & W. Ziller, *Naturally reductive metrics and Einstein metrics on compact Lie groups*, Mem. Amer. Math. Soc. No. 215, 1979.

[7] R. D'Auria, P. Fré, & P. van Nieuwenhuizen, $N = 2$ *matter coupled supergravity from compactification on a coset* $G/H$ *possessing an additional killing vector*, Phys. Lett. B **136** (1984) 347–353.

[8] J. H. Eschenburg, *New examples of manifolds with strictly positive curvature*, Invent. Math. **66** (1982) 469–480.

[9] K. Fukaya, *Collapsing Riemannian manifolds to ones of lower dimensions*, J. Differential Geometry **25** (1987) 139–156.

[10] M. Gromov, *Structure métrique pour les variétés riemanniennes*, Cédic Nathan, Paris, 1981.

[11] W. C. Hsiang & W. Y. Hsiang, *Some problems in differentiable transformation groups*, Proc. Conf. Transformation Groups, Springer, Berlin, 1968, 223–234.

[12] I. M. James & J. H. C. Whitehead, *Homotopy theory of sphere bundles over spheres*. I, Proc. Lond. Math. Soc. (3) **4** (1954) 196–218.

[13] G. Jensen, *Einstein metrics on principal fibre bundles*, J. Differential Geometry **8** (1973) 599–614.

[14] A. Katsuda, *A pinching problem for locally homogeneous spaces*, Lecture Notes in Math., Vol. 1201, Springer, Berlin, 1986, 150–155.

[15] S. Kobayashi, *On compact Kähler manifolds with positive definite Ricci tensor*, Ann. of Math. (2) **74** (1961) 570–574.

[16] ——, *Topology of positively pinched Kähler manifolds*, Tôhoku Math. J. **15** (1963) 121–139.

[17] N. Koiso & Y. Sakane, *Non-homogeneous Kähler-Einstein metrics on compact complex manifolds*, Lecture Notes in Math., Vol. 1201, Springer, Berlin, 1986, 165–179.

[18] M. Kreck & S. Stolz, *A diffeomorphism classification of 7-dimensional homogeneous Einstein manifolds with* $SU(3) \times SU(2) \times U(1)$ *symmetry*, Ann. of Math. **127** (1988) 373–388.

[19] M. Kreck & G. Triantafillou, *On the classification of manifolds up to finite ambiguity*, to appear.

[20] P. T. Nagy, *Non-horizontal geodesics of a Riemannian submersion*, Acta. Sci. Math. (Szeged) **45** (1983) 347–355.

[21] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966) 459–469.

[22] A. L. Oniščik, *Inclusion relations among transitive compact transformation groups*, Amer. Math. Soc. Transl. (2) **50** (1966) 5–58.

[23] ——, *Transitive compact transformation groups*, Amer. Math. Soc. Transl. (2) **55** (1966) 153–194.

[24] D. Page & C. Pope, *Inhomogeneous Einstein metrics on complex line bundles*, Classical Quantum Gravity **4** (1987) 213–225.

[25] Y. Sakane, *Examples of compact Einstein Kähler manifolds with positive Ricci tensor*, Osaka J. Math. **23** (1986) 585–616.

[26] S. Smale, *On the structure of 5-manifolds*, Ann. of Math. (2) **75** (1962) 38–46.

[27] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. **47** (1977) 269–331.

[28] G. Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with* $c_1(M) > 0$, Invent. Math. **89** (1987) 225–246.

[29] G. Tian & S. T. Yau, *Kähler-Einstein metrics on complex surfaces with* $c_1 > 0$, Comm. Math. Phys. **112** (1987) 175–203.

[30] M. Wang, *Some examples of homogeneous Einstein manifolds in dimension seven*, Duke Math. J. **49** (1982) 23–28.

[31] M. Wang & W. Ziller, *On normal homogeneous Einstein manifolds*, Ann. Sci. École Norm. Sup. (4) **18** (1985) 563–633.

[32] ——, *Existence and non-existence of homogeneous Einstein metrics*, Invent. Math. **84** (1986) 177-194.

McMaster University
University of Pennsylvania